Lecture Notes on Algebraic Combinatorics

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Preface: This lecture mainly includes the partial answers to the book "Algebraic Combinatorics: Walks, Trees, Tableaux, and More" (Author: Richard P. Stanley). On this special anniversary, we extend our warmest felicitations and heartfelt admiration as we honor Richard P. Stanley's 80th birthday.

MSC: Primary 05-01; Secondary 05A15, 05C05, 05C50, 05C81, 05E16, 05E18.

Keywords: Walks in Graphs; The Sperner Property; Young Tableaux; The Matrix-Tree Theorem.

1 Walks in Graphs

Exercise 1.1. *Method I: Given v_i and v_j, $i \neq j$, they are two vertices of complete graph K_p. Let A_l to be the number of walks of length l from v_i to v_j, and B_l to be the number of closed walks of length l form v_i to itself. So the number of all walks of length l form v_i is*

$$(p-1)^l = (p-1)A_l + B_l.$$

Further more, we have

$$\begin{aligned} B_{l+1} - A_{l+1} &= (p-1)A_l - ((p-2)A_l + B_l) \\ &= A_l - B_l = -(B_l - A_l) \\ &= (-1)^l (B_1 - A_1) = (-1)^{l+1}. \end{aligned}$$

By $B_l - A_l = (-1)^l \Rightarrow A_l = B_l - (-1)^l$, we have

$$(p-1)^l = (p-1)B_l - (p-1)(-1)^l + B_l.$$

Therefore

$$B_l = \frac{1}{p}((p-1)^l + (p-1)(-1)^l).$$

Method II: Using Recursive Relations. Given v_i and v_j, $i \neq j$, they are two vertices of complete graph K_p. Let G_n to be the number of walks of length n from v_i to v_j, and F_n to be the number of closed walks of length l form v_i to itself. So we have $F_0 = 1,\ G_0 = 0,\ F_1 = 0,\ G_1 = 1,\ F_2 = p-1,\ G_2 = p-2$. By

$$\begin{cases} F_n = (p-1)G_{n-1} \\ G_n = (p-2)G_{n-1} + F_{n-1}, \end{cases} \tag{1}$$

We have

$$\begin{cases} F_{n-1} = (p-1)G_{n-2} \\ F_{n-1} = G_n - (p-2)G_{n-2}. \end{cases} \tag{2}$$

Further we have a recursive relation about G_n:

$$G_n - (p-1)G_{n-2} - (p-2)G_{n-1} = 0.$$

The above recurrence formula is linear and homogeneous. Its characteristic equation is $x^2 - (p-2)x - (p-1) = 0$. Then we have $x_1 = p-1, x_2 = -1$ and

$$G_n = c_1(p-1)^n + c_2(-1)^n,$$

where c_1, c_2 are undetermined coefficients. By $G_0 = 0, G_1 = 1$, we have $c_1 + c_2 = 0$, $(p-1)c_1 - c_2 = 1$. So we have $c_1 = \frac{1}{p}, c_2 = -\frac{1}{p}$. Therefore we have

$$G_n = \frac{1}{p}((p-1)^n - (-1)^n),$$
$$F_n = (p-1)G_{n-1} = \frac{1}{p}((p-1)^n + (-1)^n(p-1)).$$

Method III: We consider the number B_l of sequences $(i_1, i_2, ..., i_l, i_{l+1})$ of numbers $1, 2, 3, 4, ..., p$ such that $i_1 = i,\ i_{l+1} = i$, no two consecutive terms are equal. So $i_l \neq i_1$. We denoted the number A_l of sequences $(i_1, i_2, ..., i_l, i_{l+1})$ such that $i_1 = i, i_{l+1} \neq i$. On the one hand, if $i_1 = i$, i_2 has $p-1$ options and i_2 has $p-1$ options, and so on. We have $(p-1)^l = B_l + (p-1)A_l$. On the other hand, we have

$$B_{l+1} - A_{l+1} = (p-1)A_l - ((p-2)A_l + B_l) = (-1)^{l+1}.$$

Same as Method I, we have

$$B_l = \frac{1}{p}((p-1)^l + (p-1)(-1)^l).$$

Exercise 1.2. *By Corollary 1.3, we have*

$$f_G(l) = \lambda_1^l + \lambda_2^l + ... + \lambda_p^l = 8^l + 2 \cdot 3^l + 3 \cdot (-1)^l + (-6)^l + 5,$$

and $p = 15$. By Lemma 1.7, we have the eigenvalues of the graph G are follows:

$$8, 3, 3, -1, -1, -1, -6, 1, 1, 1, 1, 1, 0, 0, 0.$$

Let M to be the adjacency matrix of graph G, so $M+I$ is the adjacency matrix of graph G'. Therefore, the eigenvalues of the graph G' are follows:

$$9, 4, 4, 0, 0, 0, -5, 2, 2, 2, 2, 2, 1, 1, 1,$$

and

$$f_{G'}(l) = 9^l + 2 \cdot 4^l + (-5)^l + 5 \cdot 2^l + 3.$$

Exercise 1.3. *Method I (Generating Function): By the definition of a bipartite graph G with vertex (A, B). If l is odd, the walks of length l in G from a vertex in A will terminate the vertex in B, vice versa. So the closed walks of length l (odd) in G is*

$$f_G(l) = \lambda_1^l + \lambda_2^l + ... + \lambda_p^l = 0,$$

where $\lambda_1, \lambda_2, ..., \lambda_p$ is the eigenvalues of graph G.

Let $f(n)$ to be the number of closed walks of length n in graph G, so $f(2k) > 0, f(2k+1) = 0$. Suppose $\lambda_1, \lambda_2, ..., \lambda_p$ is the eigenvalues of graph G. Then

$$f(n) = \lambda_1^n + \lambda_2^n + ... + \lambda_p^n.$$

We consider the generating function of $f(n)$ as follow:

$$F(x) = \sum_{n \geqslant 0} f(n)x^n = \sum_{i=1}^{p} \sum_{n \geqslant 0} \lambda_i^n x^n = \sum_{i=1}^{p} \frac{1}{1 - \lambda_i x}.$$

If n is odd number, $f(n) = \lambda_1^n + \lambda_2^n + ... + \lambda_p^n = 0$. We have

$$\sum_{k \geqslant 0} f(2k+1)x^{2k+1} = \frac{1}{2}(\sum_{n \geqslant 0} f(n)x^n - \sum_{n \geqslant 0} f(n)(-x)^n) = 0.$$

Further we have

$$\frac{1}{2}(F(x) - F(-x)) = 0,$$

$$\sum_{i=1}^{p} \frac{1}{1 - \lambda_i x} = \sum_{i=1}^{p} \frac{1}{1 + \lambda_i x}.$$

By Lemma 1.7, we know that the $\lambda_1, \lambda_2, ..., \lambda_p$ are just a permutation of the $-\lambda_1, -\lambda_2, ..., -\lambda_p$. Therefore the nonzero eigenvalues of G come in pairs $\pm\lambda$.

For instance, $1, -1, 4, -4$ are a permutation of $-1, 1, -4, 4$.

Method II: (Adjacency Matrix) By the definition of a bipartite graph G with vertex (A, B). We have the adjacency matrix of G:

$$A(G) = \begin{pmatrix} 0 & M_{r \times s} \\ M_{s \times r}^t & 0 \end{pmatrix},$$

where there are r vertices in A, s vertices in B. Now, we consider the characteristic polynomial

$$f(x) = det\left(\begin{pmatrix} 0 & M_{r \times s} \\ M_{s \times r}^t & 0 \end{pmatrix} - xI\right) = det\begin{pmatrix} -xI & M_{r \times s} \\ M_{s \times r}^t & -xI \end{pmatrix}$$

$$
\begin{aligned}
&= det \begin{pmatrix} -xI & M_{r\times s} \\ 0 & \frac{1}{x}M^tM - xI \end{pmatrix} \\
&= |-xI_{r\times r}| \cdot |\frac{1}{x}M^tM - xI_{s\times s}| \\
&= (-x)^r \cdot \frac{1}{x^s}|M^tM - x^2 I_{s\times s}|.
\end{aligned}
$$

where I denotes the identity matrix. The $|M^tM - x^2I_{s\times s}|$ is obviously an even function. If r and s are both even or odd, there are even number of vertices in graph G. By observing the above formula, we have $f(x) = g(x^2)$ for some polynomial $g(x)$. If r is odd, s is even or r is even, s is odd, there are odd number of vertices in graph G. We have $f(x) = xg(x^2)$ for some polynomial $g(x)$.

Exercise 1.4. (a). *The complete bipartite graph K_{rs} is a special bipartite graph. If l is odd, we have $f_{K_{rs}}(l) = 0$. If l is even, the number of closed walks from u_1 of length l is given by*

$$srsr\cdots srs \cdot 1 = s^{\frac{l}{2}} r^{\frac{l}{2}-1}.$$

The number of closed walks from v_1 of length l is given by

$$rsrs\cdots rsr \cdot 1 = r^{\frac{l}{2}} s^{\frac{l}{2}-1}.$$

Thus we have

$$f_G(l) = s^{\frac{l}{2}} r^{\frac{l}{2}-1} \cdot r + r^{\frac{l}{2}} s^{\frac{l}{2}-1} \cdot s = 2s^{\frac{l}{2}} r^{\frac{l}{2}}.$$

(b) *By the result of Exercise 3 and (a), we have*

$$
\begin{aligned}
f_G(l) &= \lambda_1^l + (-\lambda_1)^l + \lambda_2^l + (-\lambda_2)^l + ... + \lambda_m^l + (-\lambda_m)^l \\
&= \begin{cases} 0 & for \quad l: \ odd \\ 2(\sqrt{rs})^l = (\sqrt{rs})^l + (-\sqrt{rs})^l & for \quad l: \ even. \end{cases}
\end{aligned}
$$

Therefore, the eigenvalues of K_{rs} are $\sqrt{rs}, -\sqrt{rs}$ (with multiplicity 1) and 0 (with multiplicity $r+s-2$).

Exercise 1.5. *We consider the adjacency matrix $A(H_n)$ of bipartite graph H_n. Therefore we have*

$$
\begin{aligned}
f_{H_n}(2) &= (n-1) + (n-1) + ... + (n-1) \\
&= 2n(n-1) = (n-1)^2 + (n-1)^2 + 2n - 2 \\
&= (n-1)^2 + (n-1)^2 + 2(n-1).
\end{aligned}
$$

The above formula holds for any $n \geq 1$. So we have

$$
\begin{aligned}
f_{H_n}(2) &= \lambda_1^2 + \lambda_2^2 + \cdots \lambda_n^2 + (-\lambda_1)^2 + (-\lambda_2)^2 + \cdots + (-\lambda_n)^2 \\
&= (n-1)^2 + (1-n)^2 + 1^2 + (-1)^2 + 1^2 + (-1)^2 + ... + 1^2 + (-1)^2.
\end{aligned}
$$

By the Exercise 3, the eigenvalues of H_n are 1 (with multiplicity $n-1$), -1 (with multiplicity $n-1$), $n-1$ (with multiplicity 1) and $1-n$ (with multiplicity 1).

We consider the adjacency matrix $A(H_n)$ of bipartite graph H_n. We have

$$A(H_n) = \begin{pmatrix} 0 & J-I \\ J-I & 0 \end{pmatrix}.$$

$$A(H_n)^2 = \begin{pmatrix} (J-I)^2 & 0 \\ 0 & (J-I)^2 \end{pmatrix} = \begin{pmatrix} (n-2)J+I & 0 \\ 0 & (n-2)J+I \end{pmatrix}.$$

By Exercise 3, we have the characteristic polynomial

$$\begin{aligned} f(x) &= (-x)^r \cdot \frac{1}{x^s}|M^tM - x^2 I_{s\times s}| \\ &= (-1)^n |M^tM - x^2 I_{n\times n}|, \end{aligned}$$

where $M = J - I$ and $M^tM = M^2 = (n-2)J + I$. Therefore,

$$\begin{aligned} f(x) &= (-1)^n |(n-2)J + I - x^2 I_{n\times n}| \\ &= (-1)^n \begin{bmatrix} n-1-x^2 & n-2 & ... & n-2 \\ n-2 & n-1-x^2 & ... & n-2 \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ n-2 & n-2 & ... & n-1-x^2 \end{bmatrix} \\ &= (-1)^n \begin{bmatrix} n-1-x^2+(n-1)(n-2) & n-2 & ... & n-2 \\ n-1-x^2+(n-1)(n-2) & n-1-x^2 & ... & n-2 \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ n-1-x^2+(n-1)(n-2) & n-2 & ... & n-1-x^2 \end{bmatrix} \\ &= (-1)^n \begin{bmatrix} (n-1)^2-x^2 & n-2 & ... & n-2 \\ 0 & 1-x^2 & ... & 0 \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ 0 & 0 & ... & 1-x^2 \end{bmatrix}. \end{aligned}$$

Form $f(x) = 0$, we have the eigenvalues of H_n are 1 (with multiplicity $n-1$), -1 (with multiplicity $n-1$), $n-1$ (with multiplicity 1) and $1-n$ (with multiplicity 1).

Exercise 1.6. *(a). First count the number of sequences $V_{i_0}, V_{i_1}, ..., V_{i_l}$ for which there exists a closed walk with vertices $v_0, v_1, ..., v_l = v_0$ (in that order) such that $v_j \in V_{i_j}$. By Corollary 1.6, the number above is*

$$n^{l-1} \cdot \frac{1}{p}((p-1)^l + (p-1)(-1)^l).$$

By $|V_1| = |V_2| = ... = |V_{l-1}| = n$, we have

$$\begin{aligned} f_{K_{n,p}}(l) &= np \cdot n^{l-1} \cdot \frac{1}{p}((p-1)^l + (p-1)(-1)^l) \\ &= n^l((p-1)^l + (p-1)(-1)^l) \\ &= (np-n)^l + (p-1)(-n)^l. \end{aligned}$$

Remark: Given a closed walk in a complete graph, the direction of the closed walk is fixed.

(b). By Corollary 1.3, we have

$$f_{K_{n,p}}(l) = \lambda_1^l + \lambda_2^l + ... + \lambda_{np}^l = (np-n)^l + (p-1)(-n)^l.$$

By the Lemma 1.7, the eigenvalues of $K_{n,p}$ are given by $np-n$ (with multiplicity 1), $-n$ (with multiplicity $p-1$), and 0 (with multiplicity $np-p$).

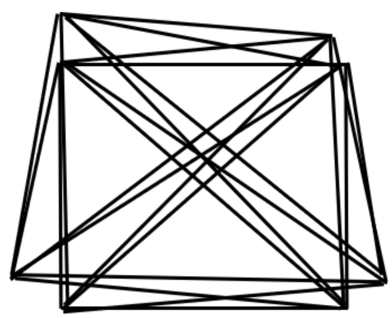

Figure 1:

Exercise 1.7. *Let G be any finite simple graph, with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_p$. The number of closed walk in G of length l is*

$$f_G(l) = \lambda_1^l + \lambda_2^l + ... + \lambda_p^l.$$

In graph $G(n)$, the number of a closed walk in G form v of length l will increase by n^{l-1} times. So we have

$$\begin{aligned} f_{G(n)}(l) &= n \cdot n^{l-1} \cdot f_G(l) \\ &= n^l(\lambda_1^l + \lambda_2^l + ... + \lambda_p^l) \\ &= (n\lambda_1)^l + (n\lambda_2)^l + ... + (n\lambda_p)^l. \end{aligned}$$

The eigenvalues of $G(n)$ are given by $n\lambda_1, n\lambda_2, ..., n\lambda_p$ and 0 (with multiplicity $np-p$).

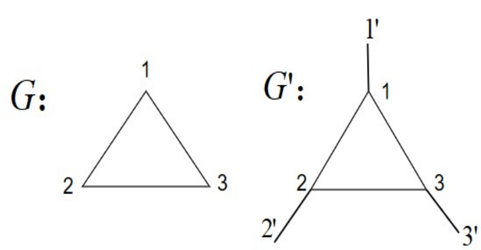

Figure 2:

Exercise 1.8. *For instance, in Fig.2, we have*

$$A(G) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

$$A(G') = \begin{pmatrix} A(G) & I \\ I & 0 \end{pmatrix}_{6\times 6}.$$

In general, we have

$$A(G') = \begin{pmatrix} A(G)_{p\times p} & I \\ I & 0 \end{pmatrix}_{2p\times 2p}.$$

where I denotes the identity matrix of $p \times p$, $A(G')$ is the matrix of $2p \times 2p$. By

$$\begin{aligned} |xI - A(G')| &= \begin{bmatrix} xI - A(G)_{p\times p} & -I \\ -I & xI \end{bmatrix}_{2p\times 2p} \\ &= \begin{bmatrix} (x - \frac{1}{x})I - A(G)_{p\times p} & 0 \\ -I & xI \end{bmatrix}_{2p\times 2p} \\ &= |xI| \cdot |(x - \frac{1}{x})I - A(G)| \\ &= 0. \end{aligned}$$

Let $t = x - \frac{1}{x}$, suppose the eigenvalues of $A(G)$ are $\lambda_1, \lambda_2, ..., \lambda_p$. Thus we have $|tI - A(G)| = c(t - \lambda_1)(t - \lambda_2)...(t - \lambda_p)$, where c independence t. By $x - \frac{1}{x} = \lambda_i$, $i = 1, 2, ..., p$, we have

$$x = \frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4}}{2}.$$

Figure 3:

Exercise 1.9. *(a). By*

$$(A(G)^l)_{ij} = \sum_{k=1}^{p} c_k(i,j)\lambda_k^l,$$

and Corollary 1.2, we have $c_k(i,i) = u_{ik} \cdot u_{ik} = u_{ik}^2 \geqslant 0$.

(b). For example, graph G in Fig.3, we have

$$A(G) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By $|\lambda E - A| = \lambda^2 - 1$, we have $\lambda = \pm 1$. Thus we have the orthonormal eigenvectors

$$v_1' = \left| \begin{matrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{matrix} \right|, v_2' = \left| \begin{matrix} \frac{1}{\sqrt{2}} \\ \frac{1}{-\sqrt{2}} \end{matrix} \right|.$$

Let
$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{-2}} \end{pmatrix}, \lambda_1 = 1, \lambda_2 = -1,$$
we have
$$A(G)^l = U \begin{pmatrix} \lambda_1^l & 0 \\ 0 & \lambda_2^l \end{pmatrix} U^{-1} = \begin{pmatrix} \frac{1}{2}\lambda_1^l + \frac{1}{2}\lambda_2^l & \frac{1}{2}\lambda_1^l - \frac{1}{2}\lambda_2^l \\ \frac{1}{2}\lambda_1^l - \frac{1}{2}\lambda_2^l & \frac{1}{2}\lambda_1^l + \frac{1}{2}\lambda_2^l \end{pmatrix}.$$
So $(A(G)^l)_{12} = \frac{1}{2}\lambda_1^l - \frac{1}{2}\lambda_2^l$. *If* $i \neq j$, *we have* $c_k(i,j) < 0$.

Exercise 1.10. *Because* G^* *is the graph with the same vertex set as* G, *and the definition of* $\eta(u,v)$, *we have* $A(G^*) = A(G)^2$ *For instance,*
$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad A(G^*) = A(G)^2 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}.$$
If the eigenvalues of $A(G)$ *are given by* $\lambda_1, \lambda_2, ..., \lambda_p$, *then the eigenvalues of* $A(G^*)$ *are given by* $\lambda_1^2, \lambda_2^2, ..., \lambda_p^2$.

Exercise 1.11. *By* $K_{21}^0 = J_{21}$ *(all 1's matrix), we can adjust the position of vertices appropriately. So the matrix of* $K_{21}^0 - K_{18}^0$ *as follow:*
$$A(\Gamma) = \begin{pmatrix} 0 & 0 & ... & 0 & 1 & 1 & 1 \\ 0 & 0 & ... & 0 & 1 & 1 & 1 \\ . & . & ... & . & . & . & . \\ . & . & ... & . & . & . & . \\ . & . & ... & . & . & . & . \\ 0 & 0 & ... & 0 & 1 & 1 & 1 \\ 1 & 1 & ... & 1 & 1 & 1 & 1 \\ 1 & 1 & ... & 1 & 1 & 1 & 1 \\ 1 & 1 & ... & 1 & 1 & 1 & 1 \end{pmatrix}.$$
Thus rank$A(\Gamma) = 2$. *Since a* 21×21 *matrix of rank* 2 *has at last* $21 - 2 = 19$ *eigenvalues equal to 0, we conclude that* $A(\Gamma)$ *has at least* 19 *eigenvalues equal to* 0. *Set the other two eigenvalues as* λ_1, λ_2. *Then* $\lambda_1 + \lambda_2 = tr(A(\Gamma)) = 3$. *By*
$$A(\Gamma)^2 = \begin{pmatrix} 3 & 3 & ... & 3 & 3 & 3 & 3 \\ 3 & 3 & ... & 3 & 3 & 3 & 3 \\ . & . & ... & . & . & . & . \\ . & . & ... & . & . & . & . \\ . & . & ... & . & . & . & . \\ 3 & 3 & ... & 3 & 3 & 3 & 3 \\ 3 & 3 & ... & 3 & 21 & 21 & 21 \\ 3 & 3 & ... & 3 & 21 & 21 & 21 \\ 3 & 3 & ... & 3 & 21 & 21 & 21 \end{pmatrix},$$

we have $rankA(\Gamma)^2 = 2$. *Therefore the eigenvalues of* $A(\Gamma)^2$ *are* λ_1^2, λ_2^2 *and* 0 *(with multiplicity* 19*). Further we have*

$$\lambda_1^2 + \lambda_2^2 = tr(A(\Gamma)^2) = 3 \times 18 + 21 \times 3 = 117.$$

By

$$\begin{cases} \lambda_1 + \lambda_2 = 3 \\ \lambda_1^2 + \lambda_2^2 = 117 \end{cases}, \tag{3}$$

we have $\lambda_1 = 9$ *or* $-6, \lambda_2 = -6$ *or* 9. *The eigenvalues of* $A(\Gamma)$ *are given by* $9, -6$ *and* 0 *(with multiplicity* 19*). Moreover, the number of closed walks in* Γ *of length* $l \geqslant 1$ *is* $C(l) = 9^l + (-6)^l$.

Exercise 1.12. *(a). Let* G *be a finite graph and let* $\triangle$ *be the maximum degree of any vertex of* G. *Let* λ_1 *be the largest eigenvalue of* $A(G)$. *We estimate the number of closed walks of length* l. *The number of walks form* v_i *in* G *length* l *is* $\leqslant \triangle^{l-1}$. *Because there are* p *vertices in* G, *The number of closed walks in* G *length* l *is* $\leqslant p \cdot \triangle^{l-1}$. *Note that we have* $p > \triangle$. *So*

$$f_G(l) = \lambda_1^l + \lambda_2^l + ... + \lambda_p^l \leqslant p \cdot \triangle^{l-1}.$$

Now, if $\lambda_1 > \triangle$, *and* $l \to \infty$, *this is contradictory. Therefore we have* $\lambda_1 \leqslant \triangle$.

(b). We consider $A(G)^2$, *let* λ^2 *be the largest eigenvalue of* $A(G)^2$. *Let* $\triangle'$ *be the maximum of the row sum of matrix* $A(G)^2$. *By Exercise 10, the* $A(G)^2$ *is a adjacency matrix of the graph* G^*. *So* $\triangle'$ *is the maximum degree of any vertex of* G^*. *By (a), we have* $\lambda^2 \leqslant \triangle'$. *Suppose*

$$A^2 = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}.$$

By $\triangle$ *is the maximum of the row sum of matrix* $A(G)$, *without loss of generality, let it be the first row of* A. *So there are* $\triangle$ *1 in the first row. The first column is also. Therefore the first element of the first column of* $A(G)^2$ *is* $\triangle$. *Because* G *has a total of* q *edges, the sum of degree of all vertex of* G *is* $2q$. *(Handshaking Theorem). Therefore, in addition to the first column, there are* $2q - \triangle$ *1's in matrix* $A(G)$. *So*

$$\triangle' \leqslant \triangle + 2q - \triangle = 2q.$$

Further we have $\lambda^2 \leqslant 2q$, $\lambda_1 \leqslant |\lambda| \leqslant \sqrt{2q}$.

Exercise 1.13. *By the given condition, there exists a positive integer* $\ell \geq 1$ *such that for any pair of vertices* u, v, *the number of walks of length* ℓ *between them is odd. Therefore, we have:*

$$A(G)^\ell = J = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \mod 2.$$

Since $A(G)^\ell$ is not of full rank, it follows that $A(G)$ is not of full rank as well. Hence, there exists a non-zero vector α such that $A(G)\alpha = 0$. If $\alpha \pmod 2$ contains an odd number of 1s, then

$$A(G)^\ell \alpha \pmod 2 = J\alpha \pmod 2 \neq 0,$$

which is a contradiction. Therefore, $\alpha \pmod 2$must contain an even number of 1s. Moreover, every vertex in G is connected to an even number of vertices corresponding to the 1s in $\alpha \pmod 2$ (otherwise, it would contradict $A(G)\alpha = 0 \mod 2$). Thus, the set S is exactly the set of vertices corresponding to the 1s in $\alpha \pmod 2$.

2 Cubes and the Randon Transform

This chapter mainly uses Radon transformation to obtain the eigenvalues and eigenvectors of the n-dimensional cube C_n, and then considers the walks on C_n. However, by the relevant results of the Cartesian products of the graph, the eigenvalues and eigenvectors of C_n can be directly obtained.

Define: Given graph G_1 andG_2 with vertex set V and W, respectively. Their Cartesion product $G_1 \boxdot G_2$ is the graph with vertex set $V \times W$, where $(v, w) \sim (v', w')$ when either $v = v'$ and $w \sim w'$ or $w = w'$ and $v \sim v'$.

The adjacency matrix of the Cartesian product $G_1 \boxdot G_2$ of two graphs G_1 and G_2 is

$$A(G_1 \boxdot G_2) = A(G_1) \otimes I + I \otimes A(G_2),$$

where I is the identity matrix, and $\otimes$ is the tensor.

Let the eigenvalues of two graphs G_1 and G_2 be $\lambda_1, ..., \Lambda_p$ and μ_1, μ_q, respectively. The the eigenvalues of Cartesian products $G_1 \boxdot G_2$ are $\lambda_i + \mu_j$, where $1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q$.

Therefore, we have $C_2 := C_1 \boxdot C_1$. Similarly, we have

$$C_n = C_1 \boxdot C_1 \boxdot C_1 \boxdot \cdots \boxdot C_1,$$

where there are n C_1. The eigenvalues of C_1 are $-1, 1$. So the eigenvalues of C_n are $n - 2i$ with multiplicity $\binom{n}{i}$, $i = 0, 1, ..., n$.

The above results can also be referred to in the book: A E. Brouwer and W.H. Haemers, Spectra of Graphs, Springer. This book can also serve as an in-depth understanding of the content of the first three chapters.

Stanley uses Radon transformation to solve problems in the book, mainly to illustrate the power of the algebraic method of Radon transformation, as also noted in his book.

Exercise 2.1. *(a). If the coin have heads up, we denote it as 0. If the coin have tails up, we denote it as 1. Then all coins have heads up is $(0, 0, ..., 0)$, and tails up is $(1, 1, ..., 1)$. Therefore, the state of the front and back of the coin can be seen as the vertices of n-Cube C_n. We regard the flip process of the coin as the walks in graph C_n.*

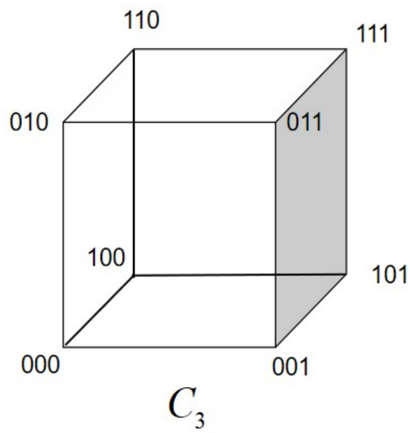

Figure 4:

By Corollary 2.5, the number of closed walks of form $u = (0, 0, ..., 0)$ *in* C_n *length* l *is*

$$(A^l)_{uu} = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (n - 2i)^l.$$

So the probability that all coins will have heads up is

$$P = \frac{(A^l)_{uu}}{n^l} = \frac{\sum_{i=0}^{n} \binom{n}{i} (n - 2i)^l}{2^n n^l}.$$

(b) We need to seek the number of walks of length l *from* $(0, 0, ..., 0)$ *to* $(1, 1, ..., 1)$*. By Corollary 2.5,* $u = (0, 0, ..., 0), v = (1, 1, ..., 1), \omega(u + v) = n$*, we have*

$$(A^l)_{uv} = \frac{1}{2^n} \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{n-n}{i-j} (n - 2i)^l = \frac{1}{2^n} \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n - 2i)^l.$$

Therefore the probability that all coins will have tails up is

$$P = \frac{(A^l)_{uv}}{n^l} = \frac{\sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n - 2i)^l}{2^n n^l}.$$

(c). Now we turn over two coins at a time. We regard it as flipping one and then flipping another, but the second flip coin cannot be the first flip coin. For instance, $n = 3$*, we obtain* $(1, 1, 0), (0, 1, 1), (1, 0, 1)$ *by turn over two steps. There are two ways to reach the goal. So the adjacency matrix of the new graph is*

$$\frac{1}{2}(A(C_n)^2 - nI),$$

where I *denotes the identity matrix. Because the eigenvalues of* $A(C_n)$ *are* $\lambda_u = n - 2\omega(u), u \in Z_2^n$*, we have the eigenvalues of* $\frac{1}{2}(A(C_n)^2 - nI)$ *are* $\frac{\lambda_u^2 - n}{2}, u \in Z_2^n$*. Therefore the number of* *closed walks of length l from $(0,0,...,0)$ is*

$$\begin{aligned}(A^l)_{uu} &= \frac{1}{2^n}\sum_{u\in Z_2^n}\Big(\frac{\lambda_u^2-nI}{2}\Big)^l \\ &= \frac{1}{2^{n+l}}\sum_{u\in Z_2^n}(\lambda_u^2-n)^l \\ &= \frac{1}{2^{n+l}}\sum_{i=0}^{n}\binom{n}{i}[(n-2i)^2-n]^l.\end{aligned}$$

So the probability is

$$P=\frac{(A^l)_{uu}}{\binom{n}{2}^l}=\frac{\sum_{i=0}^n\binom{n}{i}[(n-2i)^2-n]^l}{2^{n+l}\binom{n}{2}^l}.$$

Exercise 2.2. *Omit.*

Exercise 2.3. *Omit.*

Exercise 2.4. *By $\omega(u+v)=2$, same as Exercise 1(c), the adjacency matrix of the graph G is*

$$\frac{1}{2}(A(C_n)^2-nI).$$

Because the eigenvalues of $A(C_n)$ are $\lambda_u=n-2\omega(u), u\in Z_2^n$, we have the eigenvalues of G are

$$\frac{\lambda_u^2-n}{2}=\frac{(n-2\omega(u))^2-n}{2}, u\in Z_2^n.$$

Therefore we have $\binom{n}{i}$ eigenvalues $\frac{(n-2i)^2-n}{2}$. For $i=0,1,2,...,\lfloor\frac{n}{2}\rfloor$, there are $\binom{n}{i}+\binom{n}{n-i}$ eigenvalues $\frac{(n-2i)^2-n}{2}$ for G.

Exercise 2.5. *(a).*

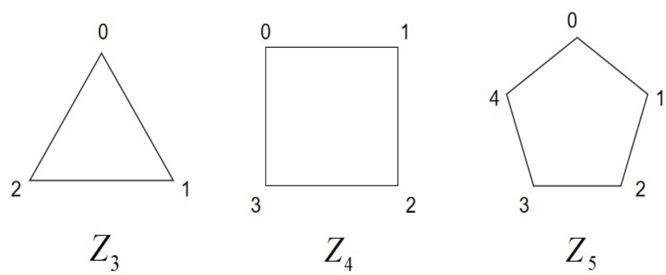

Figure 5:

(b). The dimension of $\mathcal{V}$ is n. The basis, denoted B_1, has elements f_u defined as follows:

$$f_u(v)=\delta_{uv}, u\in Z_n.$$

Since any $g \in \nu$ satisfies

$$g = \sum_{u \in Z_n} g(u) f_u.$$

Hence B_1 spans $\mathcal{V}$. Because $k_1 f_{u_1} + k_2 f_{u_2} + ... + k_n f_{u_n} = 0$, both sides act on $u_1, u_2, ..., u_n$ at the same time, we have $k_1 = k_2 = ... = k_n = 0$. Hence they are linearly independent as desired. We have $dim\mathcal{V} = n$.

(c). In the complex vector space, we will use an inner product on $\mathcal{V}$ defined by

$$< f, g >= \sum_{u \in Z_n} f(u)\overline{g(u)}.$$

In order to show that B is a basis, it suffices to show that B is linearly independent. (Since $\#B = dim\mathcal{V} = n$). In fact, we will show that the elements of B are orthogonal. By

$$\begin{aligned}
< \chi_u, \chi_v > &= \sum_{w \in Z_n} \chi_u(w)\overline{\chi_v(w)} = \sum_{w \in Z_n} \zeta^{u \cdot w} \cdot \overline{\zeta^{v \cdot w}} \\
&= \sum_{w \in Z_n} e^{\frac{2\pi i}{n} \cdot uw} \cdot \overline{e^{\frac{2\pi i}{n} \cdot vw}} = \sum_{w \in Z_n} e^{\frac{2\pi i}{n} \cdot uw} e^{-\frac{2\pi i}{n} \cdot vw} \\
&= \sum_{w \in Z_n} e^{\frac{2\pi i}{n}(u-v)w} = \sum_{w \in Z_n} \zeta^{(u-v)w},
\end{aligned}$$

If $u - v = 0$, i.e., $u = v$, we have $< \chi_u, \chi_v >= n$. If $u - v \neq 0$, i.e., $u \neq v$, for any $y \in Z_n, (y \neq 0)$, we have

$$\begin{aligned}
\sum_{w \in Z_n} \zeta^{y \cdot w} &= 1 + \zeta^{y \cdot 1} + \zeta^{y \cdot 2} + ... + \zeta^{y \cdot (n-1)} \\
&= 0.
\end{aligned}$$

The second "=" because (let $x = \zeta^y$)

$$x^n - 1 = (x-1)(1 + x + x^2 + ... + x^{n-1}).$$

So the elements of B are orthogonal. Hence they are independent as desired.

(d). By

$$\begin{aligned}
\Phi_\Gamma \chi_u(v) &= \sum_{w \in \Gamma} \chi_u(v + w) \\
&= \sum_{w \in \Gamma} \zeta^{u \cdot (v+w)} \\
&= \sum_{w \in \Gamma} \zeta^{u \cdot w} \cdot \zeta^{u \cdot v} \\
&= \sum_{w \in \Gamma} \zeta^{u \cdot w} \cdot \chi_u(v),
\end{aligned}$$

we have $\Phi_\Gamma \chi_u = \sum_{w \in \Gamma} \zeta^{u \cdot w} \cdot \chi_u$. Thus the eigenvector of Φ_Γ are the functions χ_u, with corresponding eigenvalue $\lambda_u = \sum_{w \in \Gamma} \zeta^{u \cdot w}$.

(e). The proof process is the same as Lemma 2.3. By

$$\begin{aligned}\Phi_\Delta f_u(v) &= \sum_{w\in\Delta} f_u(v+w)\\ &= f_u(v+1)+f_u(v+n-1) = \delta_{u(v+1)}+\delta_{u(v+n-1)}\\ &= \delta_{(u+n-1)v}+\delta_{(u+1)v}\\ &= f_{u+1}(v)+f_{u+n-1}(v),\end{aligned}$$

we have $\Phi_\Delta f_u = f_{u+1}+f_{u+n-1}$. *For* $f_0, f_1, ..., f_{n-1}$, *we obtain*

$$\Phi_\Delta(f_0,f_1,...,f_{n-1}) = (f_0,f_1,...,f_{n-1})\begin{pmatrix} 0 & 1 & 0 & ... & 0 & 0 & 1\\ 1 & 0 & 1 & ... & 0 & 0 & 0\\ 0 & 1 & 0 & ... & 0 & 0 & 0\\ . & . & . & ... & . & . & .\\ . & . & . & ... & . & . & .\\ . & . & . & ... & . & . & .\\ 0 & 0 & . & ... & 1 & 0 & 1\\ 1 & 0 & . & ... & 0 & 1 & 0 \end{pmatrix}.$$

This matrix is exactly the adjacency matrix $A(Z_n)$ *of graph* Z_n.

(f). The matrix $[\Phi_\Delta]$ *of* Φ_Δ *with respect to the basis* F *is just* $A(Z_n)$, *the adjacency matrix of* Z_n. *By (d), the eigenvalues of* $A(Z_n)$ *are* $\lambda_u = \sum_{w\in\Delta}\zeta^{uw}$. *Because* $\Delta = \{1, n-1\}$, *we have*

$$\begin{aligned}\lambda_u &= \zeta^u + \zeta^{u(n-1)}\\ &= e^{\frac{2\pi i}{n}u} + e^{\frac{2\pi i}{n}(un-u)}\\ &= e^{\frac{2\pi i}{n}u} + e^{-\frac{2\pi i}{n}u}\\ &= cos\frac{2\pi u}{n} + i\cdot sin\frac{2\pi u}{n} + cos\frac{2\pi u}{n} - i\cdot sin\frac{2\pi u}{n}\\ &= 2cos\frac{2\pi u}{n},\end{aligned}$$

where $u = 0,1,2,...,n-1$. *So the eigenvalues of* $A(Z_n)$ *are* $2cos\frac{2\pi j}{n}, 0 \leqslant j \leqslant n-1$. *Now, we show that the corresponding eigenvectors of* $\lambda_u = 2cos\frac{2\pi u}{n}$ *are* E_u *(as below). Similarly, by the proof process of Corollary 2.4, we have* $g = \sum_v g(v)f_v$ *for any* $g \in \mathcal{V}$. *If* $g = \chi_u$, *we have*

$$\chi_u = \sum_{v\in Z_n}\chi_u(v)f_v = \sum_{v\in Z_n}\zeta^{uv}f_v.$$

Therefore

$$E_u = \sum_{v\in Z_n}\zeta^{uv}\cdot v.$$

The expansion of eigenvectors of Φ_Δ *in terms of the* f_v*'s has the same coefficients as the expansion of the eigenvectors of* $A(Z_n)$ *in terms of the* v*'s.*

(g) The number of closed walks of length l beginning with 0 is

$$A(Z_n)_{00} = \frac{1}{n} f_G(l) = \frac{1}{n} \sum_{u=0}^{n-1} (2cos\frac{2\pi u}{n})^l.$$

But without using trigonometric functions, complex exponentials, etc.

For $n = 4$. Method I:

Let $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *we have* $B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E$, $B^3 = B, B^4 = E, B^5 = B, ...$ *So*

$$A(Z_n) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} B & B \\ B & B \end{pmatrix}.$$

Further we have

$$\begin{aligned} A(Z_n)^l &= \begin{pmatrix} B & B \\ B & B \end{pmatrix}^l \\ &= \begin{pmatrix} 2^{l-1}B^l & 2^{l-1}B^l \\ 2^{l-1}B^l & 2^{l-1}B^l \end{pmatrix} \\ &= \begin{cases} \begin{pmatrix} 2^{l-1}E & 2^{l-1}E \\ 2^{l-1}E & 2^{l-1}E \end{pmatrix} & l \text{ is even} \\ \begin{pmatrix} 2^{l-1}B & 2^{l-1}B \\ 2^{l-1}B & 2^{l-1}B \end{pmatrix} & l \text{ is odd} \end{cases} \\ &= \begin{cases} \begin{pmatrix} 2^{l-1} & 0 & 2^{l-1} & 0 \\ 0 & 2^{l-1} & 0 & 2^{l-1} \\ 2^{l-1} & 0 & 2^{l-1} & 0 \\ 0 & 2^{l-1} & 0 & 2^{l-1} \end{pmatrix} & l \text{ is even;} \\ \begin{pmatrix} 0 & 2^{l-1} & 0 & 2^{l-1} \\ 2^{l-1} & 0 & 2^{l-1} & 0 \\ 0 & 2^{l-1} & 0 & 2^{l-1} \\ 2^{l-1} & 0 & 2^{l-1} & 0 \end{pmatrix} & l \text{ is odd.} \end{cases} \end{aligned}$$

The number of closed walks of length l beginning with 0 is $\begin{cases} 2^{l-1} & l \text{ is even;} \\ 0 & l \text{ is odd.} \end{cases}$

Method II: We find that graph Z_4 is the same as graph Z_2^2, i.e., 2-Cube. So

$$(A^l)uu = \frac{1}{4} \sum_{i=0}^{2} \binom{2}{i} (2-2i)^l = \frac{1}{4}(2^l - (-2)^l) = \begin{cases} 2^{l-1} & l \text{ is even;} \\ 0 & l \text{ is odd.} \end{cases}$$

For $n = 6$. Method I:

$$f = \frac{1}{6}\left[(2cos\frac{\pi}{3})^l + (2cos\frac{2\pi}{3})^l + (2cos\frac{3\pi}{3})^l + (2cos\frac{4\pi}{3})^l + (2cos\frac{5\pi}{3})^l + (2cos2\pi)^l\right]$$

$$= \frac{1}{6} \cdot 2^l \cdot \left[(\frac{1}{2})^l + (-\frac{1}{2})^l + (-1)^l + (-\frac{1}{2})^l + (\frac{1}{2})^l + 1^l \right]$$

$$= \begin{cases} \frac{1}{3} \cdot 2^l + \frac{2}{3} & l \text{ is even;} \\ 0 & l \text{ is odd.} \end{cases}$$

Method II: If l is odd, the number of closed walks of length l clearly is 0. If l is even, let F_l to be the number of closed walks of length l form some vertex i to itself, and G_l^{i+2} to be the number of walks of length l form i to $i+2$. In this Exercise, let $i = 0$. Because the degree for any vertices is 2, we have

$$2^l = F_l + 2G_l^{i+2}.$$

(by l is even, beginning with 0, end up $0, 2, 4$).

In addition, we have

$$F_l = 2G_{l-1}^{i+1} \quad (1).$$

(the number of walks form 0 to 1 of length $l-1$ + the number of walks form 0 to 5 of length $l-1$).

Moreover, we have

$$G_{l-1}^{i+1} = F_{l-2} + G_{l-2}^{i+2} \quad (2).$$

(the number of walks form 0 to 1 of length $l-1$=the number of walks form 0 to 0 of length $l-2$ + the number of walks form 0 to 2 of length $l-2$).

Bring formula (2) into formula (1), we have

$$F_l = 2F_{l-2} + 2G_{l-2}^{i+2},$$

$$F_l - F_{l-2} = F_{l-2} + 2G_{l-2}^{i+2} = 2^{l-2}.$$

By $F_0 = 1, F_2 = 2$, we have

$$F_l = \frac{1}{3} \cdot 2^l + \frac{2}{3}.$$

($F_2 - F_0 = 2^0, F_4 - F_2 = 2^2, F_6 - F_4 = 2^4, ..., F_l - F_{l-2} = 2^{l-2}$, we have $F_l = 2 + 2^2 + 2^4 + \cdots + 2^{l-2} = \frac{1}{3} \cdot 2^l + \frac{2}{3}$.).

(h). By $n \geq 2$, We can draw the graphs for $n = 2, 3, 4, 5, 6$.

For $n \geqslant 5$, the adjacency matrix of the $Z_n^{(2)}$ is $[A(Z_n)^2 + A(Z_n) - 2I]$, where I denotes the identity matrix. (Because walks twice in $Z_n^{(2)}$ will result in closed walks). So the eigenvalues of $Z_n^{(2)}$ are

$$4cos^2\frac{2\pi j}{n} + 2cos\frac{2\pi j}{n} - 2, \ 0 \leqslant j \leqslant n-1.$$

Further more,

$$F_l = \frac{1}{n}\Big(\sum_{j=0}^{n-1}(4cos^2\frac{2\pi j}{n} + 2cos\frac{2\pi j}{n} - 2)^l\Big).$$

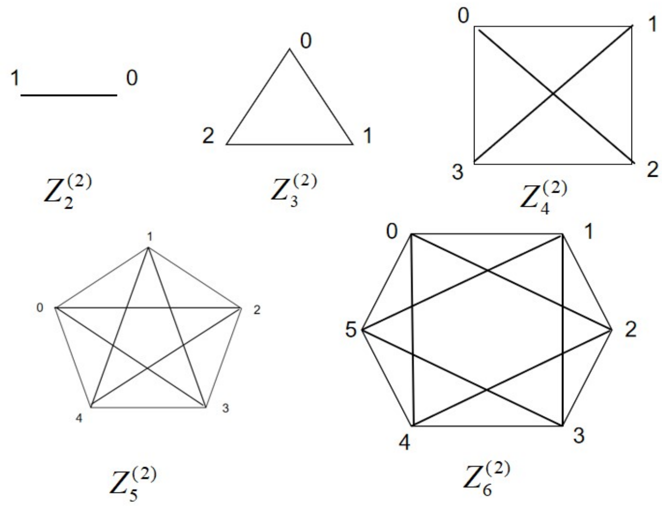

Figure 6:

For $n = 2$, *we have* $F_l = \begin{cases} 0 & l \text{ is even;} \\ 1 & l \text{ is odd.} \end{cases}$

For $n = 3$, *we have*

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, |\lambda E - A| = (\lambda + 1)(\lambda^2 - \lambda - 2) = (\lambda + 1)^2(\lambda - 2)$$

$$F_l = \frac{1}{3}((-1)^l) + (-1)^l + 2^l) = \frac{1}{3}(2(-1)^l + 2^l).$$

For $n = 4$, *we have*

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

$$|\lambda E - A| = \begin{bmatrix} \lambda & -1 & -1 & -1 \\ -1 & \lambda & -1 & -1 \\ -1 & -1 & \lambda & -1 \\ -1 & -1 & -1 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda & -1 & -1 & -1 \\ -1 & \lambda & -1 & -1 \\ 0 & -\lambda - 1 & \lambda + 1 & 0 \\ 0 & -\lambda - 1 & 0 & \lambda + 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & -3 & -1 & -1 \\ -1 & \lambda - 2 & -1 & -1 \\ 0 & 0 & \lambda + 1 & 0 \\ 0 & 0 & 0 & \lambda + 1 \end{bmatrix}$$

$$= \lambda(\lambda-2)(\lambda+1)^2 + (-3)(\lambda+1)^2$$
$$= (\lambda+1)^3(\lambda-3).$$

So we have $F_4 = \frac{1}{4}(3 \cdot (-1)^l + 3^l)$.

Exercise 2.6. *The proof process is similar to Lemma 2.3 and Corollary 2.4. We define that*

$$\widetilde{\Delta} = \Delta \cup (1,1,1,...,1).$$

Let $[\Phi_{\widetilde{\triangle}}]$ *denote the matrix of the linear transformation* $\Phi_{\widetilde{\triangle}} : \mathcal{V} \to \mathcal{V}$.

Suppose $v \in Z_2^n$, *we have*

$$\Phi_{\widetilde{\triangle}} f_u(v) = \sum_{w \in \widetilde{\triangle}} f_u(v+w) = \sum_{w \in \widetilde{\triangle}} f_{u+w}(v).$$

So

$$\Phi_{\widetilde{\triangle}} f_u = \sum_{w \in \widetilde{\triangle}} f_{u+w}.$$

Above formula say that $(u,v)-$*entry (short for* $(f_u, f_v)-$*entry) of the matrix* $[\Phi_{\widetilde{\triangle}}]$ *is given by*

$$(\Phi_{\widetilde{\triangle}})_{uv} = \begin{cases} 1 & if\ u+v \in \widetilde{\triangle} \\ 0 & otherwise. \end{cases}$$

Now $u+v \in \widetilde{\triangle}$ *if and only if* u *and* v *differ in exactly one coordinate or* $u+v = (1,1,1,...,1)$. *This is just the condition for* uv *to be an edge of* $\widetilde{C}_n$.

We have $[\Phi_{\widetilde{\triangle}}] = A(\widetilde{C}_n)$. *By*

$$\lambda_u = \sum_{v \in \widetilde{\triangle}} (-1)^{uv} = n - 2\omega(u) + (-1)^{\omega(u)},$$

($\omega(u)$ *is called the Hamming weight). we have*

$$(\widetilde{A}^l)_{uu} = \frac{1}{2^n} \sum_{i=0}^{n} \left(\binom{n}{i} ((n-2i) + (-1)^i)^l \right).$$

3 Random Walks

Exercise 3.1. *...*

Exercise 3.2. *...*

Exercise 3.3. *(a). ($\Rightarrow$), if all coefficients of $f(x) = P(x)Q(x)$ are nonnegative, then $f(a) = \sum a_n a^n, \forall a > 0, a_n \geqslant 0$. So $f(a) > 0$ and $P(a)Q(a) > 0$. There does not exist a real number $a > 0$ such that $P(a) = 0$.*

($\Leftarrow$),....

(b)...

Exercise 3.4. *Let M' be the probability matrix. The graph C_n is regular of degree n. Start at the vertex $(0, 0, ..., 0)$, after each unit of time, either stay where you are with probability p, the probability of stepping to a particular neighboring vertex is $\frac{1-p}{n}$. So we have*

$$M'(C_n) = \frac{1-p}{n} A(C_n) + pI,$$

where I denotes the identity matrix. The eigenvalues of $M'(C_n)$ are

$$\lambda(M'(C_n)) = \frac{1-p}{n}(n - 2\omega(u)) + p.$$

Therefore

$$\begin{aligned} P(l) &= \frac{1}{2^n} \sum_{u \in Z_2^n} \Big(\frac{1-p}{n}(n - 2\omega(u)) + p\Big)^l \\ &= \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \Big(\frac{1-p}{n}(n - 2i) + p\Big)^l. \end{aligned}$$

Exercise 3.5. *...*

Exercise 3.6. *In Figure 7: By the Theorem 3.4 and Example 3.5, this is easily to obtain the*

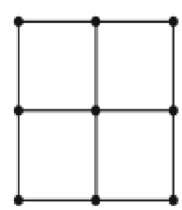

Figure 7:

$H(u, v)$. We can use the software "Matlab", "Mathematica" and "Maple" compute the access time $H(u, v)$. For example, we use "Maple" to compute the follow (Figure 8):

Exercise 3.7. *(a). We just take counterexample to verify.*

For Fig.9, we have $H(u, w) = 1$ and $H(w, u) = \frac{1}{2} \times 1 + \frac{1}{2}(2 + H(w, u))$. So $H(w, u) = 3$. Then we need not have $H(u, w) = H(w, u)$.

If G ia also assumed to be regular, we have $H(u, v) = H(v, u)$.

(b)....

```
> with(LinearAlgebra) :
> M := Matrix([[0, 1/2, 0, 1/2, 0, 0, 0, 0, 0], [1/3, 0, 1/3, 0, 1/3, 0, 0, 0, 0], [0, 1/2, 0, 0, 0, 1/2, 0, 0, 0], [1/3, 0, 0, 0, 1/3, 0, 1/3,
    0, 0], [0, 1/4, 0, 1/4, 0, 1/4, 0, 1/4, 0], [0, 0, 1/3, 0, 1/3, 0, 0, 0, 1/3], [0, 0, 0, 1/2, 0, 0, 0, 1/2, 0], [0, 0, 0, 0, 1/3, 0, 1/3, 0,
    1/3], [0, 0, 0, 0, 0, 1/2, 0, 1/2, 0]]) :
> I8 := Matrix(8, shape = identity) :
> DeleteRow(M, 6) :
> MV6 := DeleteColumn(%, 6) :
> T6 := M[[1, 2, 3, 4, 5, 7, 8, 9], [6]] :
> simplify(MatrixInverse(I8 − MV6)) : MatrixMatrixMultiply(%, %) :
> MatrixMatrixMultiply(%, T6);
```

$$\begin{bmatrix} 12 \\ 10 \\ 6 \\ 12 \\ 9 \\ 12 \\ 10 \\ 6 \end{bmatrix} \qquad (1)$$

```
>
```

Figure 8:

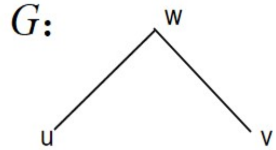

Figure 9:

Exercise 3.8. *In complete graph K_n, let the vertex set of K_n be $V = \{v_1, v_2, ..., v_n\}$. It is easy to see that $H(u_1, u_n) = H(u_2, u_n) = \cdots = H(u_{n-1}, u_n)$. So*

$$\begin{aligned} H(u_1, u_n) &= \frac{1}{n-1} \times 1 + \frac{1}{n-1}(1 + H(u_2, u_n)) + \frac{1}{n-1}(1 + H(u_3, u_n)) + ... \\ &\quad ... + \frac{1}{n-1}(1 + H(u_{n-1}, u_n)) \\ &= \frac{1}{n-1} \times 1 + \frac{n-2}{n-1}(1 + H(u_1, u_n)). \end{aligned}$$

Therefore, we have $H(u_1, u_n) = n - 1$.

Exercise 3.9. *(1). Induction method. For $n = 1$, we have $H(v_1, v_1) = (1-1)^2 = 0$. For $n = 2$, we have $H(v_1, v_2) = (2-1)^2 = 1$. Suppose we have $H(v_1, v_{n-1}) = (n-2)^2$. Now we consider the $H(v_1, v_n)$. So*

$$H(v_1, v_n) = H(v_1, v_{n-1}) + H(v_{n-1}, v_n) = (n-2)^2 + H(v_{n-1}, v_n).$$

By

$$\begin{cases} H(v_{n-1}, v_n) = \frac{1}{2} + \frac{1}{2}(1 + H(v_{n-2}, v_n)) \\ H(v_{n-2}, v_n) = \frac{1}{2}(1 + H(v_{n-1}, v_n)) + \frac{1}{2}(1 + H(v_{n-3}, v_n)) \\ H(v_{n-3}, v_n) = \frac{1}{2}(1 + H(v_{n-2}, v_n)) + \frac{1}{2}(1 + H(v_{n-4}, v_n)) \\ . \\ . \\ . \\ H(v_2, v_n) = \frac{1}{2}(1 + H(v_3, v_n)) + \frac{1}{2}(1 + H(v_1, v_n)) \\ H(v_1, v_n) = 1 + H(v_2, v_n). \end{cases}$$

Further more,

$$\begin{cases} H(v_{n-1}, v_n) = 1 + \frac{1}{2}H(v_{n-2}, v_n) \\ H(v_{n-2}, v_n) = 1 + \frac{1}{2}H(v_{n-1}, v_n) + \frac{1}{2}H(v_{n-3}, v_n) \\ H(v_{n-3}, v_n) = 1 + \frac{1}{2}H(v_{n-2}, v_n) + \frac{1}{2}H(v_{n-4}, v_n) \\ . \\ . \\ . \\ H(v_2, v_n) = 1 + \frac{1}{2}H(v_3, v_n) + \frac{1}{2}H(v_1, v_n) \\ H(v_1, v_n) = 1 + H(v_2, v_n). \end{cases}$$

Therefore we have

$$\frac{1}{2}H(v_{n-1},v_n)=(n-1)+\frac{1}{2}.$$

Thus $H(v_{n-1},v_n)=2n-3$. *So we have*

$$H(v_1,v_n)=(n-2)^2+2n-3=(n-1)^2.$$

(2). For $i\neq j$, *if* $i<j$, *we have* $H(v_1,v_j)=H(v_1,v_i)+H(v_i,v_j)$. *Then we have*

$$H(v_i,v_j)=(j-1)^2-(i-1)^2=j^2-2j-i^2+2i.$$

If $i>j$, *by the symmetry of graph and (1), we have* $H(v_n,v_j)=H(v_n,v_i)+H(v_i,v_j)$. *So* $H(v_i,v_j)=(n-j)^2-(n-i)^2$.

Therefore we have

$$\begin{cases} H(v_i,v_j)=(j-1)^2-(i-1)^2 & if\ i<j \\ H(v_i,v_j)=(n-j)^2-(n-i)^2 & if\ i>j. \end{cases}$$

(3). If we also have an edge between v_1 *and* v_n. *For* $H(v_1,v_n)$, *we have*

$$\begin{cases} H(v_1,v_n)=\frac{1}{2}\cdot 1+\frac{1}{2}(1+H(v_2,v_n)) \\ H(v_2,v_n)=\frac{1}{2}(1+H(v_1,v_n))+\frac{1}{2}(1+H(v_3,v_n)) \\ H(v_3,v_n)=1+\frac{1}{2}H(v_3,v_n)+\frac{1}{2}H(v_4,v_n) \\ . \\ . \\ . \\ H(v_{n-2},v_n)=1+\frac{1}{2}H(v_{n-3},v_n)+\frac{1}{2}H(v_{n-1},v_n) \\ H(v_{n-1},v_n)=\frac{1}{2}\times 1+\frac{1}{2}(1+H(v_{n-2},v_n)=1+\frac{1}{2}H(v_{n-2},v_n). \end{cases}$$

Thus we have

$$\begin{aligned} &H(v_2,v_n)+H(v_{n-2},v_n)+H(v_{n-1},v_n) \\ &\quad=n-2+\frac{1}{2}H(v_1,v_n)+\frac{1}{2}H(v_2,v_n) \\ &\quad+\frac{1}{2}H(v_{n-2},v_n)+\frac{1}{2}H(v_{n-1},v_n)+\frac{1}{2}H(v_{n-2},v_n). \end{aligned}$$

By the symmetry of graph, we have $H(v_1,v_n)=H(v_{n-1},v_n)$. *So* $\frac{1}{2}H(v_2,v_n)=n-2$ *and*

$$H(v_1,v_n)=1+n-2=n-1.$$

(4). By (3), we have

$$\begin{cases} H(v_1, v_n) = n-1 \\ H(v_2, v_n) = 2n - 2^2 \\ H(v_3, v_n) = 3n - 3^2 \\ \dots. \end{cases}$$

By the symmetry of graph, recursive, we have

$$\begin{cases} H(v_i, v_n) = in - i^2 \quad if\ i \leqslant \lfloor \frac{n}{2} \rfloor; \\ H(v_i, v_n) = (n-i)n - (n-i)^2 \quad if\ i > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

Now, let m be the length form i to j, the length of the shortest path. For $i \neq j$, we have $m \leqslant \lfloor \frac{n}{2} \rfloor$. So $n - m \geqslant \lfloor \frac{n}{2} \rfloor$. Therefore we have

$$H(v_i, v_j) = H(v_{n-m}, v_n) = mn - m^2.$$

Exercise 3.10. *Let K_{mn} be a complete bipartite graph with vertex bipartition (A_1, A_2), where $\#A_1 = m$, $\#A_2 = n$. The degree of the vertex in A_1 is n, and the degree of the vertex in A_2 is m. Suppose $u_1, u_2 \in A_1$, $u_1 \neq u_2$ and $v_1, v_2 \in A_2$ and $v_1 \neq v_2$. So*

$$\begin{cases} H(u_1, v_1) = \frac{1}{n} \times 1 + \frac{n-1}{n}(1 + H(v_2, v_1)) = 1 + \frac{n-1}{n} H(v_2, v_1) \\ H(v_2, v_1) = \frac{1}{m}(1 + H(u_1, v_1)) \times m = 1 + H(u_1, v_1). \end{cases}$$

Further we have

$$\begin{cases} H(u_1, v_1) = 2n - 1; \quad (*1) \\ H(v_2, v_1) = 2n. \quad (*2) \end{cases}$$

By

$$\begin{cases} H(v_2, u_1) = \frac{1}{m} \times 1 + \frac{m-1}{m}(1 + H(u_2, u_1)) = 1 + \frac{m-1}{m} H(u_2, u_1) \\ H(u_2, u_1) = \frac{1}{n}(1 + H(v_2, u_1)) \times n = 1 + H(v_2, u_1). \end{cases}$$

We have

$$\begin{cases} H(v_2, u_1) = 2m - 1 \quad (*3) \\ H(u_2, u_1) = 2m. \quad (*4) \end{cases}$$

For $(*1)(*2)(*3)(*4)$, this are two inequivalent cases.

Exercise 3.11. ......

Exercise 3.12. *The probability is* $(M[v]^n)_{uw}$ *that a random walk starting* u *takes to reach* w *and without passing through point* v *in* n *steps. Therefore*

$$p_n = \sum_{w \neq v} M[v]_{uw}^{n-1} \cdot \frac{\mu_{wv}}{d_w} = (M[v]^{n-1}T[v])_u,$$

where $T[v]$ *is the column vector of length* $p-1$ *whose rows are indexed by the vertices* $w \neq v$*, and* $T[v]_w = \frac{\mu_{wv}}{d_w}$*. So*

$$H_k(u,v) = \sum_{n \geqslant 1} \binom{n}{k} (M[v]^{n-1}T[v])_u$$

and the generating function

$$\begin{aligned}\sum_{k \geqslant 0} H_k(u,v)x^k &= \left(\sum_{k \geqslant 0} \sum_{n \geqslant 1} \binom{n}{k} x^k M[v]^{n-1} T[v] \right)_u \\ &= \left(\sum_{n \geqslant 1} \sum_{k \geqslant 0} \binom{n}{k} x^k M[v]^{n-1} T[v] \right)_u .\end{aligned}$$

By

$$\sum_{k \geqslant 0} \binom{n}{k} x^k = \binom{n}{0} x^0 + \binom{n}{1} x^1 + \dots + \binom{n}{1} x^n = (1+x)^n,$$

we have

$$\begin{aligned}\sum_{k \geqslant 0} H_k(u,v)x^k &= \left(\sum_{n \geqslant 1} (1+x)^n M[v]^{n-1} T[v] \right)_u \\ &= (((1+x)M[v]^0 + (1+x)^2 M[v]^1 + \dots)T[v])_u \\ &= (1+x)((I_{p-1} - (x+1)M[v])^{-1} T[v])_u.\end{aligned}$$

Exercise 3.13. ......

4 The Sperner Property

Exercise 4.1. *Hasses diagram of the 16 nonisomorphic four-element posets (Figure 10):*

Hasses diagram of the 63 nonisomorphic four-element posets (Figure 11, 12, 13, 14):

Exercise 4.2. (a) *By* $f : P \to P$ *is an order-preserving bijection in a finite poset* P*, the* f *is a permutation on finite set* P*. So we have* $f^n = id$ *for some* n*, where* id *is an identity mapping. Therefore* $f^{-1} = f^{n-1}$*. By* f *is an order-preserving bijection, the* f^{n-1} *is also. So the* f^{-1} *is an order-preserving bijection.*

(b) *if* P *is infinite, the conclusion may not be valid.*

Counterexample 1: Let $P = \mathbb{Z} \cup x$ *(*$\mathbb{Z}$ *denote the set of all integers),* $x < 0$ *and* x *is not comparable to all* $n < 0$*. Let* $f(x) = x$ *and* $f(n) = n+1$ *for* $n \in \mathbb{Z}$*. So we have* $x < 0$ *under the map* f*, but* x *and* -1 *is not comparable under the map* f^{-1}*. So* f^{-1} *is not order-preserving.*

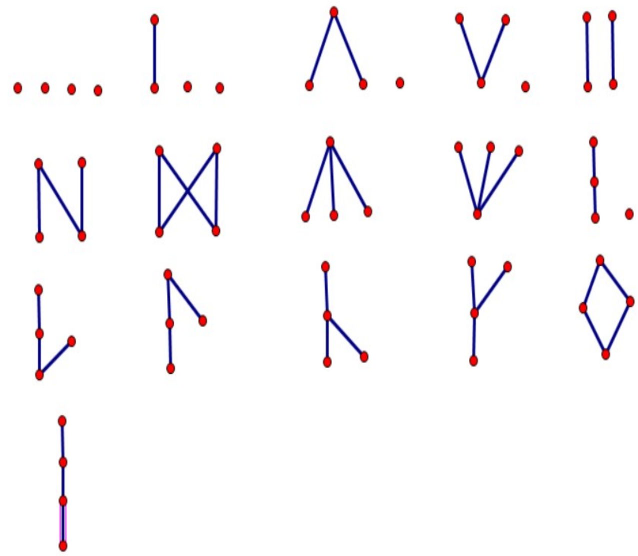

Figure 10:

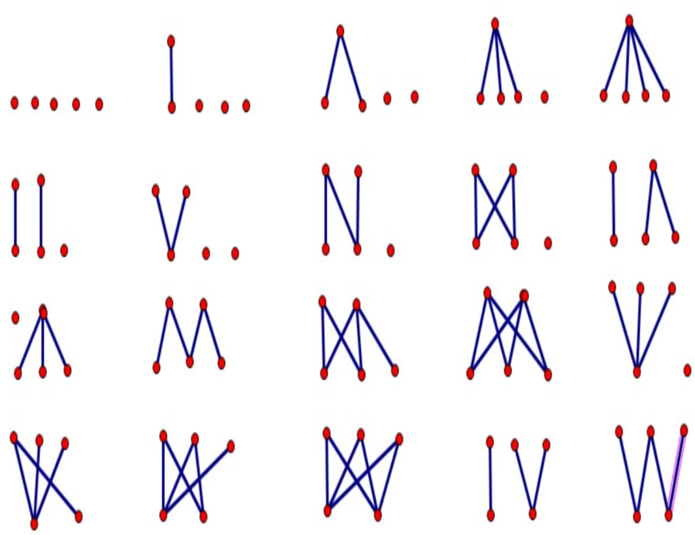

Figure 11:

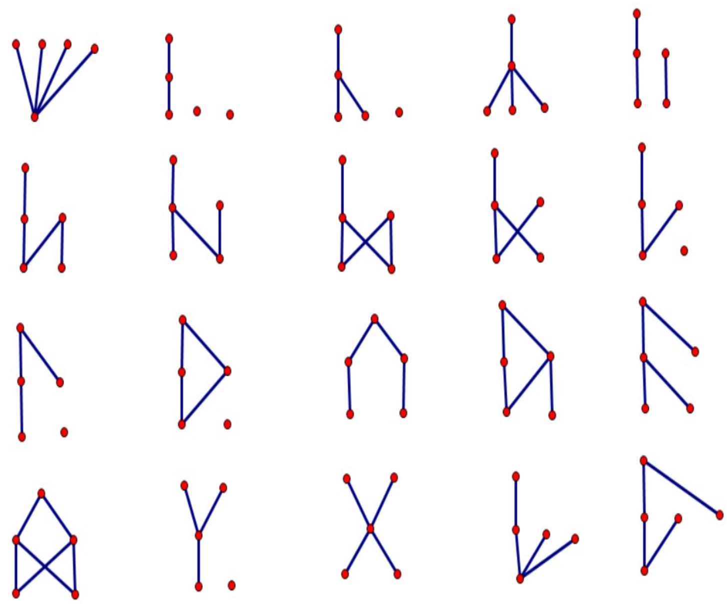

Figure 12:

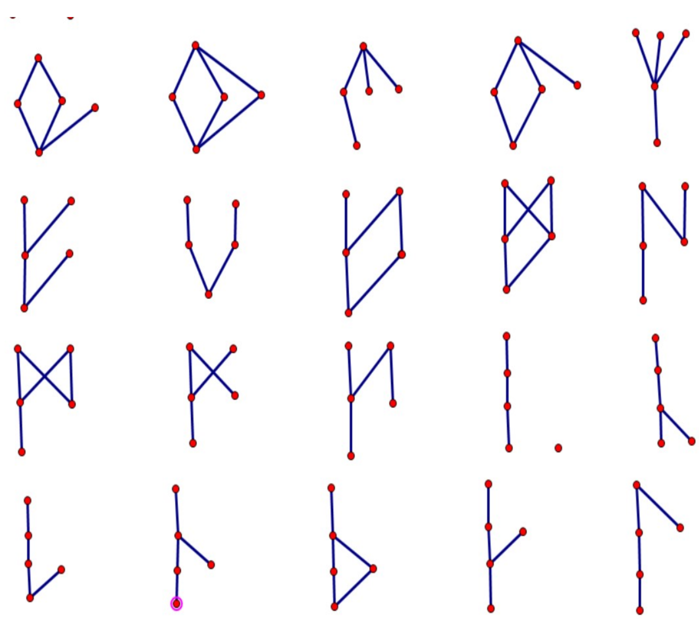

Figure 13:

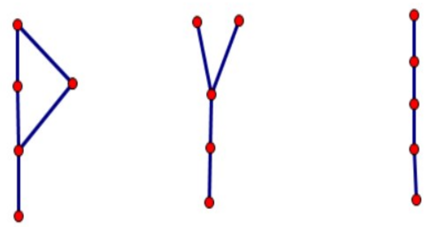

Figure 14:

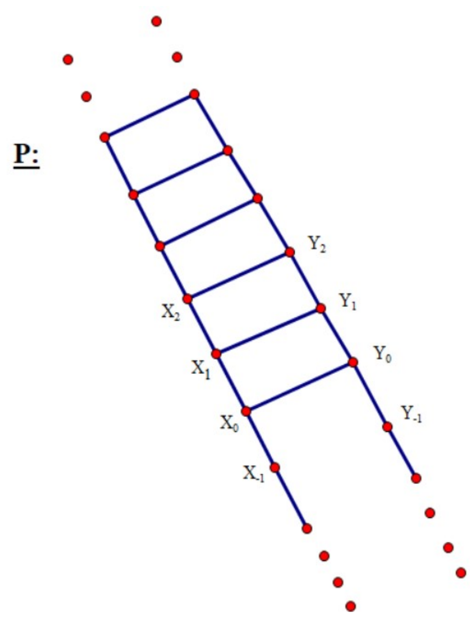

Figure 15:

Counterexample 2: In Figure 15. Let $\varphi : P \to P$. For any $i \in \mathbb{Z}$, we have $x_i \to x_{i+1}$, $y_i \to y_{i+1}$. So φ^{-1} is not an order-preserving map. For example $y_0 \geqslant x_0$, but y_{-1} and x_{-1} is not comparable under the map φ^{-1}.

Exercise 4.3. *Method I: Let $F(q) = p_0+p_1q+p_2q^2+...+p_nq^n$, $G(q) = r_0+r_1q+r_2q^2+...+r_mq^m$. By $G(q), F(q)$ is symmetric unimodal polynomials with nonnegative real coefficients. So we have*

$$\begin{cases} p_i, r_i \geqslant 0 \quad p_i = p_{n-i} \quad r_i = r_{m-i} \quad i = 0, 1, 2, ..., n \setminus m \\ p_0 \leqslant p_1 \leqslant ... \leqslant p_{\left\lfloor \frac{n}{2} \right\rfloor} \geqslant p_{\left\lfloor \frac{n}{2} \right\rfloor +1} \geqslant ... \geqslant p_n \\ r_0 \leqslant r_1 \leqslant ... \leqslant r_{\left\lfloor \frac{m}{2} \right\rfloor} \geqslant r_{\left\lfloor \frac{m}{2} \right\rfloor +1} \geqslant ... \geqslant r_n. \end{cases}$$

Therefore

$$\begin{aligned} F(q)G(q) = & p_0r_0 + (p_0r_1 + p_1r_0)q + (p_0r_2 + p_1r_1 + p_2r_0)q^2 + ...+ \\ & \sum_{i+j} p_ir_jq^{i+j} + ... + (p_mr_{m-1} + p_{m-1}r_m)q^{m+n-1} + (p_nr_m)q^{m+n}. \end{aligned}$$

By observing the above equation, we have

$$p_0r_0 = p_nr_m, \quad p_0r_1 + p_1r_0 = p_nr_{m-1} + p_{n-1}r_m, ...$$

By $p_i = p_{n-i}, r_i = r_{m-i}$, for example, the coefficient before q^2 is $p_0r_2+p_1r_1+p_2r_0 = p_{n-0}r_{m-2}+ p_{n-1}r_{m-1}+p_{n-2}r_{m-0}$. The right end of the equality is exactly the coefficient in front of q^{m+n-2}. For the same reason, the coefficient before q^s is exactly equal to the coefficient before q^{m+n-s}. Therefore $F(q)G(q)$ is symmetric polynomials.

Let

$$F(q)G(q) = t_0 + t_1q^1 + t_2q^2 + ... + t_{m+n-1}q^{m+n-1} + t_{m+n}q^{m+n}.$$

By

$$\begin{cases} p_0r_0 \leqslant p_0r_1 \leqslant p_0r_1 + p_1r_0 \ \Rightarrow \ t_0 \leqslant t_1 \\ p_1r_0 + p_0r_1 \leqslant p_0r_2 + p_1r_1 \leqslant p_0r_2 + p_1r_1 + p_2r_0 \ \Rightarrow \ t_1 \leqslant t_2 \\ ... \\ ... \\ p_nr_m \leqslant p_nr_{m-1} \leqslant p_nr_{m-1} + p_{n-1}r_m \ \Rightarrow \ t_{m+n} \leqslant t_{m+n-1} \\ p_nr_{m-1} + p_{n-1}r_m \leqslant p_nr_{m-2} + p_{n-1}r_{m-1} \leqslant p_nr_{m-2} + p_{n-1}r_{m-1} + p_{n-2}r_m \\ \qquad\qquad \Rightarrow \ t_{m+n-1} \leqslant t_{m+n-2} \\ ... \\ ... \end{cases}$$

and $F(q)G(q)$is symmetric polynomials, we can know that $F(q)G(q)$ is unimodal polynomials.

Method II: Let $F(q) = p_0 + p_1q + p_2q^2 + ... + p_nq^n$, $G(q) = r_0 + r_1q + r_2q^2 + ... + r_mq^m$. By $G(q), F(q)$ is symmetric unimodal polynomials with nonnegative real coefficients. So we have

$$q^nF\left(\frac{1}{q}\right) = F(q), \qquad q^mG\left(\frac{1}{q}\right) = G(q).$$

Therefore, we have

$$q^{m+n}F\left(\frac{1}{q}\right)G\left(\frac{1}{q}\right) = F(q)G(q).$$

So $F(q)G(q)$ is also symmetric.

By $F(q)$ is symmetric and unimodal, $F(q)$ can be expressed as

$$F(q) = p_0(1+q+\cdots+q^n) + p_1'(q+q^2+\cdots+q^{n-1}) + \cdots + \begin{cases} 0 & \textit{if } n \textit{ is odd,} \\ p_{\frac{n}{2}}q^{\frac{n}{2}} & \textit{if } n \textit{ is even.} \end{cases}$$

Define $\frac{n-1}{2}$ (n: odd) or $\frac{n}{2}$ (n: even) as the center *of $F(q)$.*

For example,

$$\begin{aligned} 3+5q+5q^2+7q^3+7q^4+5q^5+5q^6+3q^7 &= 3(1+q+q^2+q^3+q^4+q^5+q^6+q^7) \\ &+ 2(q+q^2+q^3+q^4+q^5+q^6) + 0(q^2+q^3+q^4+q^5) + 2(q^3+q^4) + 0. \end{aligned}$$

Similarly, $G(q)$ can be expressed as

$$G(q) = r_0(1+q+\cdots+q^m) + r_1'(q+q^2+\cdots+q^{m-1}) + \cdots + \begin{cases} 0 & \textit{if } m \textit{ is odd,} \\ r_{\frac{m}{2}}q^{\frac{m}{2}} & \textit{if } m \textit{ is even.} \end{cases}$$

Define $\frac{m-1}{2}$ (m: odd) or $\frac{m}{2}$ (m: even) as the center *of $G(q)$.*

Now, we can check that

$$(q^i + q^{i+1} + \cdots + q^{i+j})(q^k + q^{k+1} + \cdots + q^{k+l})$$

is symmetric and unimodal. Furthermore, all products of this kind have the same center. Therefore, $F(q)G(q)$ is the sum of several products of the above type. So $F(q)G(q)$ is symmetric and unimodal.

Exercise 4.4. *(a). By $q = 2, n = 3$, let $\alpha = (a_1, a_2, a_3) \in V = F_2^3$. We have $a_1, a_2, a_3 \in 0, 1 \ \ (1+1=0)$. Let*

$$\begin{aligned} &\alpha_1 = (0,0,0),\ \alpha_2 = (1,0,0),\ \alpha_3 = (0,1,0),\ \alpha_4 = (0,0,1), \\ &\alpha_5 = (1,1,0),\ \alpha_6 = (0,1,1),\ \alpha_7 = (1,0,1),\ \alpha_8 = (1,1,1), \end{aligned}$$

therefore $\alpha_1 \sim \alpha_8$ are all the elements in V. Now we consider the subspace of V:
0-dimensional: $\alpha_1 = (0,0,0)$;
1-dimensional: $<\alpha_2>=\{\alpha_1,\alpha_2\}$, $<\alpha_3>=\{\alpha_1,\alpha_3\}$,..., $<\alpha_8>=\{\alpha_1,\alpha_8\}$;
2-dimensional:

$$\begin{aligned} &<\alpha_2,\alpha_3>=\{\alpha_1,\alpha_2,\alpha_3,\alpha_5\},\ <\alpha_2,\alpha_4>=\{\alpha_1,\alpha_2,\alpha_4,\alpha_7\}, \\ &<\alpha_3,\alpha_4>=\{\alpha_1,\alpha_3,\alpha_4,\alpha_6\},\ <\alpha_2,\alpha_6>=\{\alpha_1,\alpha_2,\alpha_6,\alpha_8\}, \\ &<\alpha_3,\alpha_7>=\{\alpha_1,\alpha_3,\alpha_7,\alpha_8\},\ <\alpha_4,\alpha_5>=\{\alpha_1,\alpha_4,\alpha_5,\alpha_8\}, \\ &<\alpha_5,\alpha_6>=<\alpha_5,\alpha_7>=<\alpha_6,\alpha_7>=\{\alpha_1,\alpha_5,\alpha_6,\alpha_7\}; \end{aligned}$$

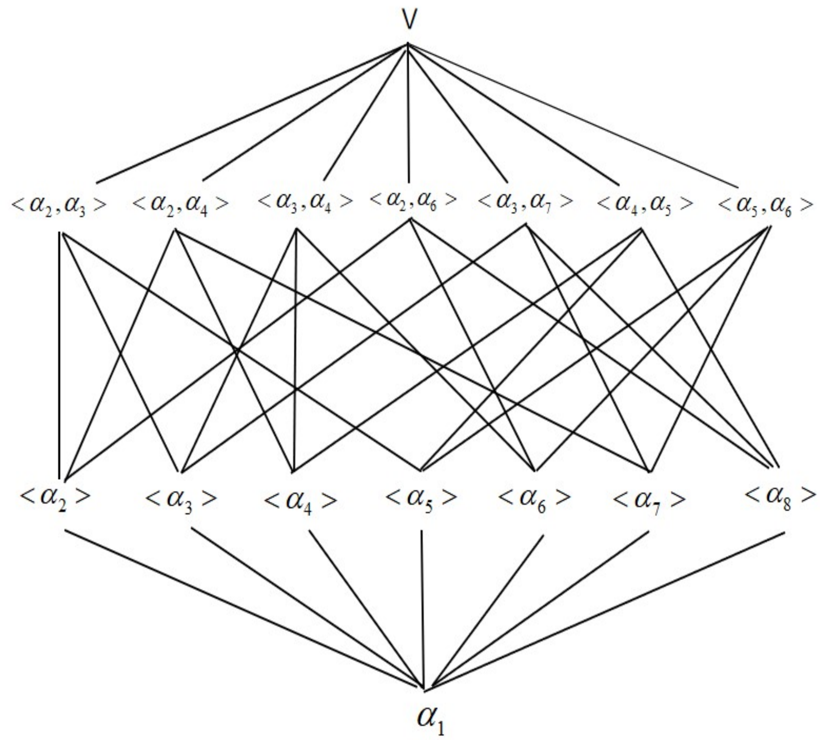

Figure 16:

3-dimensional: $V = <\alpha_2, \alpha_3, \alpha_4>$.

So the Hasse diagram of $B_3(2)$ *is Figure 16:*

(b). (Use double counting). We count the number of linearly independent k*-tuples* $(v_1, v_2, ..., v_k)$ *of the vector space* F_q^n*. For a* k*-dimensional subspace* $L(\alpha_1, \alpha_2, ..., \alpha_k)$ *of* F_q^n*, we count the number of* $\alpha_1, \alpha_2, ..., \alpha_k$ *such that* $\alpha_1, \alpha_2, ..., \alpha_k$ *linear independence. On the one hand, for* α_1*, there are* $q^n - 1$ *options. For* α_2*, there are* $q^n - q$ *options. For* α_3*, there are* $q^n - q^2$ *options. .... For* α_k*, there are* $q^n - q^{k-1}$ *options. There are a total of* $(q^n - 1)(q^n - q)...(q^n - q^{k-1})$ *options. On the other hand, let's first choose a* k*-dimensional subspace* W*, and there are* $\begin{bmatrix} n \\ k \end{bmatrix}$ *options. After that there are* $q^k - 1$ *options to choose the* $\alpha_1 \in W$*. For* $\alpha_2 \in W$*, there are* $q^k - q$ *options..... For* $\alpha_k \in W$*, there are* $q^k - q^{k-1}$ *options. Therefore, we have*

$$\begin{bmatrix} n \\ k \end{bmatrix} (q^k - 1)(q^k - q)...(q^k - q^{k-1}) = (q^n - 1)(q^n - q)...(q^n - q^{k-1}).$$

Because $\begin{bmatrix} n \\ k \end{bmatrix}$ *is the number of elements with rank* k *in* $B_n(q)$*. So*

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} &= \frac{q^{k-1} \cdot q^{k-2} \cdot ... \cdot q(q^{n-k+1} - 1)(q^{n-k+2} - 1)...(q^n - 1)}{q^{k-1} \cdot q^{k-2} \cdot ... \cdot q(q - 1)(q^2 - 1)...(q^k - 1)} \\ &= \frac{(q^{n-k+1} - 1)(q^{n-k+2} - 1)...(q^n - 1)}{(q - 1)(q^2 - 1)...(q^k - 1)} \end{aligned}$$

$$= \frac{(1-q^n)...(1-q^{n-k+1})}{(1-q)(1-q^2)...(1-q^k)}.$$

(c). In the poset $B_n(q)$ (graded of rank n), the number of element of rank 0 is 1, denote $p_0 = 1$. By (b), for $1 \leqslant k \leqslant n$, we know that the number of elements of rank k are

$$p_k = \frac{(q^n-1)(q^{n-1}-1)...(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)...(q-1)}.$$

So we have $p_n = 1 = p_0$ if $k = n$. When $1 \leqslant k \leqslant n$, we just need to prove $p_k = p_{n-k}$. By

$$p_k = \frac{(q^n-1)(q^{n-1}-1)...(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)...(q-1)}, p_{n-k} = \frac{(q^n-1)(q^{n-1}-1)...(q^{k+1}-1)}{(q^{n-k}-1)(q^{n-k-1}-1)...(q-1)}.$$

we just need to prove

$$\frac{(q^n-1)(q^{n-1}-1)...(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)...(q-1)} = \frac{(q^n-1)(q^{n-1}-1)...(q^{k+1}-1)}{(q^{n-k}-1)(q^{n-k-1}-1)...(q-1)}.$$

After that we just need to prove

$$\begin{aligned}&(q^n-1)(q^{n-1}-1)...(q^{n-k+1}-1)(q^{n-k}-1)(q^{n-k-1}-1)...(q-1)\\&=(q^k-1)(q^{k-1}-1)...(q-1)(q^n-1)(q^{n-1}-1)...(q^{k+1}-1).\end{aligned}$$

The above formula is clearly hold, so $p_k = p_{n-k}$, $0 \leqslant k \leqslant n$. Therefore $B_n(q)$ is rank-symmetric.

(d).

See Figure 17:

Method I: We first to prove that every element $x \in B_n(q)_k$ is covered by $\boldsymbol{(n-k)} = 1 + q + q^2 + ... + q^{n-k-1}$ elements. In layer P_k, element x is k-dimensional, the element y that cover element x is $k+1$-dimensional and it contains the elements x. The space V is n-dimensional. In the P_{k+1}-th layer, element α_{k+1} should be selected in the $n-1$-dimension. So there are a total of $q^{n-k}-1$ options. (Subtracting 1 is because we need to remove $(0,0,0,...,0)$.) By F_q denote the finite field with q elements, we have $\frac{q^{n-k}-1}{q-1}$ options. By

$$\frac{q^{n-k}-1}{q-1} = q^{n-k-1} + q^{n-k-2} + ... + q + 1,$$

the element x is covered by $\boldsymbol{(n-k)} = 1 + q + q^2 + ... + q^{n-k+1}$ elements.

For example, for the $B_3(2)$ in (a), we have $<\alpha_2>= \{(0,0,0),(1,0,0)\}$ if $k=1$. For the remaining $n-k = 3-1 = 2$ dimensions, we can take $(1,0)(0,1)(1,1)$, i.e., $1+q = 1+2 = 3$.

For the first half, similarly, each element in layer $k-1$ is covered by $\frac{q^{n-k+1}-1}{q-1}$ elements. In layer $k-1$, there are P_{k-1} elements. Therefore, there are a total of $\frac{q^{n-k+1}-1}{q-1}P_{k-1}$ coverage relationships between the $k-1$-th and $k-1$-th layers. In layer k, there are P_k elements. These

(d) 下证 $\forall x \in B_n(q)_k$，x 被 $[n-k] = 1+q+\cdots+q^{n-k-1}$ 个元素覆盖

证 令 $x = L(v_1,\cdots,v_k)$，覆盖 x 的元素为 $k+1$ 维线性子空间.

① 先找线性无关的向量 $u_1,\cdots,u_{k+1}$ 满足 $v_1,\cdots,v_k \leftrightarrow u_1,\cdots,u_k$，且 $u_{k+1} \nrightarrow v_1,\cdots,v_k$.

u_1 有 q^k-1 种选择，u_2 有 q^k-q 种选择，$\cdots$，u_k 有 q^k-q^{k-1} 种选择. u_{k+1} 有 q^n-q^k 种选择.

则这样的 $u_1,\cdots,u_{k+1}$ 有 $(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})(q^n-q^k)$ 个.

② 下求线性无关的向量 $u_1',\cdots,u_{k+1}'$ 满足 $u_1',\cdots,u_{k+1}' \leftrightarrow u_1,\cdots,u_{k+1}$，$u_{k+1}' \nrightarrow u_1,\cdots,u_k$ 且 $u_{k+1}' \rightarrow u_1,u_2,\cdots,u_{k+1}$.

u_1' 有 q^k-1 种选择. u_2' 有 q^k-q 种选择，$\cdots$，u_k' 有 q^k-q^{k-1} 种选择. u_{k+1}' 有 $q^{k+1}-q^k$ 种选择.

则这样的 $u_1'\cdots u_{k+1}'$ 有 $(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})(q^{k+1}-q^k)$ 个.

综上，覆盖 x 的元素为 $k+1$ 维线性子空间的个数为

$$\frac{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})(q^n-q^k)}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})(q^{k+1}-q^k)} = 1+q+\cdots+q^{n-k-1}.$$

□

Figure 17:

P_k elements are symmetric for the coverage relationship. Therefore, for any element x in layer k, a total of

$$\left(\frac{q^{n-k+1}-1}{q-1}P_{k-1}\right)\frac{1}{P_k}=\frac{q^k-1}{q-1}=1+q+q^2+...+q^{k-1}$$

elements are covered by it.

Note: for

$$\left(\frac{q^{n-k+1}-1}{q-1}P_{k-1}\right)\frac{1}{P_k}=\frac{q^k-1}{q-1},$$

we just need to prove

$$\left(q^{n-k+1}-1\right)P_{k-1}=(q^k-1)P_k.$$

We just need to prove

$$(q^{n-k+1}-1)\frac{(q^n-1)\cdot...\cdot(q^{n-k+2}-1)}{(q^{k-1}-1)\cdot...\cdot(q-1)}=\frac{(q^k-1)(q^n-1)\cdot...\cdot(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdot...\cdot(q-1)}.$$

The above formula is clearly hold.

Method II: We use the method in (b): Selecting $k-1$-dimensional subspaces in a specific k-dimensional space. We have

$$\binom{\mathbf{k}}{\mathbf{k-1}}=\frac{(q^k-1)\cdot...\cdot(q^2-1)}{(q^{k-1}-1)\cdot...\cdot(q-1)}=\frac{q^k-1}{q-1}=1+q+q^2+...+q^{k-1},$$

i.e., x covers $\boldsymbol{(k)}$ elements. Similarly, we have

$$\binom{\mathbf{k+1}}{\mathbf{k}}=\frac{q^{k+1}-1}{q-1}.$$

There are $\binom{n}{k+1}$ elements with rank $k+1$. There are $\binom{n}{k}$ elements with k. Therefore

$$\binom{\mathbf{n}}{\mathbf{k+1}}\binom{\mathbf{k+1}}{\mathbf{k}}=\binom{\mathbf{n}}{\mathbf{k}}\cdot\boldsymbol{(n-k)}.$$

(The number of intermediate connections between the upper and lower layers in the Hasse diagram). We have

$$\begin{aligned}\boldsymbol{(n-k)}&=\frac{\frac{(q^n-1)\cdot...\cdot(q^{n-k}-1)}{(q^{k+1}-1)\cdot...\cdot(q-1)}\cdot\frac{q^{k+1}-1}{q-1}}{\frac{(q^n-1)\cdot...\cdot(q^{n-k+1}-1)}{(q^k-1)\cdot...\cdot(q-1)}}\\&=\frac{q^{n-k}-1}{q-1}=1+q+q^2+...+q^{n-k-1}.\end{aligned}$$

This completes the proof.

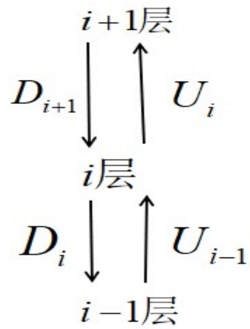

Figure 18:

(e). The notation

$$\begin{aligned}(\boldsymbol{n-k}) &= 1+q+q^2+...+q^{n-k-1}\\ (\boldsymbol{k}) &= 1+q+q^2+...+q^{k-1}.\end{aligned}$$

Suppose $x \in B_n(q)_i$. We have

$$((\boldsymbol{n-i})-(\boldsymbol{i}))I_i(x) = ((\boldsymbol{n-i})-(\boldsymbol{i}))x,$$

and

$$\begin{aligned}D_{i+1}U_i(x) &= D_{i+1}\left(\sum_{y\in B_n(q)_{i+1}\ \ y>x} y\right)\\ &= \sum_{y\in B_n(q)_{i+1}\ \ y>x} (D_{i+1}y)\\ &= \sum_{y\in B_n(q)_{i+1}\ \ y>x} \left(\sum_{z\in B_n(q)_i\ \ y>z} z\right).\end{aligned}$$

If the dimension of $|x \cap z| < i-1$, i.e., the difference between x and z is at least 2, then there is no $y \in B_n(q)_{i+1}$ such that $x < y, z < y$. Therefore, in the expansion of the above equation, the coefficient before z is 0.
If the dimension of $|x \cap z| = i-1$, i.e., the difference between x and z is only 1 item, then there is only one such y, i.e, $y = x \cup z$.
If the dimension of $|x \cap z| = i$, i.e, $x = z$, then there are a total of $(\boldsymbol{n-i})$ such y (By (d)).

Therefore we have

$$D_{i+1}U_i(x) = (\boldsymbol{n-i})x + \sum_{z\in B_n(q)_i\ \ |x\cap z|=i-1} z.$$

Similarly, we have

$$U_{i-1}D_i(x) = U_{i-1}\left(\sum_{y\in B_n(q)_{i-1}\ \ y<x} y\right)$$

$$= \sum_{y\in B_n(q)_{i-1}\ y<x} \left(\sum_{z\in B_n(q)_i\ y<z} z \right).$$

If the dimension of $|x\cap z| < i-1$, then there is no $y \in B_n(q)_{i+1}$ such that $x < y, z < y$. Therefore, in the expansion of the above equation, the coefficient before z is 0.
If the dimension of $|x\cap z| = i-1$, then there is only one such y, i.e, $y = x\cup z$.
If the dimension of $|x\cap z| = i$, i.e, $x = z$, then there are a total of $(\boldsymbol{i})$ such y (By (d)).

Therefore we have

$$U_{i-1}D_i(x) = (\boldsymbol{i})x + \sum_{z\in B_n(q)_i\ |x\cap z|=i-1} z.$$

So we obtain

$$D_{i+1}U_i(x) - U_{i-1}D_i(x) = (\boldsymbol{n-i})x - (\boldsymbol{i})x,$$
$$D_{i+1}U_i - U_{i-1}D_i = ((\boldsymbol{n-i}) - (\boldsymbol{i}))I_i.$$

This completes the proof.

(f). Let $[U_i]$ represent the matrix corresponding to U_i under bases $B_n(q)_i$ and $B_n(q)_{i+1}$. Let $[D_i]$ represent the matrix corresponding to D_i under bases $B_n(q)_i$ and $B_n(q)_{i-1}$. Suppose

$$U_i(x_1,x_2,...,x_m) = (y_1,y_2,...,y_n)\begin{pmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1m} \\ a_{21} & a_{22} & a_{23} & ... & a_{2m} \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & . \\ a_{n1} & a_{n2} & a_{n3} & ... & a_{nm} \end{pmatrix}_{n\times m},$$

where $a_{ij} \in \{0,1\},\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$. Therefore covering x_1 should be $\{a_{11}y_1, a_{21}y_2, ..., a_{n1}y_n\}$; covering x_2 should be $\{a_{12}y_1, a_{22}y_2, ..., a_{n2}y_n\}$;...; covering x_m should be $\{a_{1m}y_1, a_{2m}y_2, ..., a_{nm}y_n\}$. So we know that covering y_1 should be $\{a_{11}x_1, a_{12}x_2, ..., a_{1m}x_m\}$. By analogy, we can obtain the elements covered by $y_2, y_3, ...$ By

$$D_{i+1}(y_1,y_2,...,y_n) = (x_1,x_2,...,x_m)\begin{pmatrix} a_{11} & a_{21} & a_{31} & ... & a_{n1} \\ a_{12} & a_{22} & a_{32} & ... & a_{n2} \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & . \\ a_{1m} & a_{2m} & a_{3m} & ... & a_{nm} \end{pmatrix}_{m\times n},$$

we have $[U_i] = [D_{i+1}]$, i.e., $[U_{i-1}] = [D_i]^t$. Therefore $U_{i-1}D_i$ is a positive semi-definite matrix, it has non negative real eigenvalues. By (e), we have

$$D_{i+1}U_i - U_{i-1}D_i = ((\boldsymbol{n-i}) - (\boldsymbol{i}))I_i.$$

So The eigenvalues of $D_{i+1}U_i$ are obtained by adding $(\boldsymbol{n-i})-(\boldsymbol{i})$ to the eigenvalues of $U_{i-1}D_i$. By

$$(\boldsymbol{n-i}) = 1 + q + q^2 + ... + q^{n-i-1} = \frac{q^{n-i}-1}{q-1},$$

$$\boldsymbol{(i)} = 1 + q + q^2 + ... + q^{i-1} = \frac{q^i - 1}{q - 1},$$

when $\boldsymbol{(n-i)} - \boldsymbol{(i)} > 0$, *we have*

$$\frac{q^{n-i} - 1}{q - 1} > \frac{q^i - 1}{q - 1} \ (q > 1) \Rightarrow q^{n-i} > q^i \Rightarrow n - i > i \Rightarrow i < \frac{n}{2}.$$

Now, the eigenvalues of $D_{i+1}U_i$ *are non zero positive numbers. So Matrix* $D_{i+1}U_i$ *is invertible,* $rank(D_{i+1}U_i) = rank(U_i) = m$ *and* U_i *is column full rank. Therefore if* $i < \frac{n}{2}$*, the* U_i *is injection. Similarly, if* $i \geqslant \frac{n}{2}$*, the* U_i *surjective. Because* U_i *is an order-raising operator, by Lemma 4.5, we know that there exists an order-matching* $\mu : B_n(q)_i \to B_n(q)_{i+1}$ *when* $i < \frac{n}{2}$*. There exists an order-matching* $\mu : B_n(q)_{i+1} \to B_n(q)_i$ *when* $i \geqslant \frac{n}{2}$*. By Proposition 4.4, we know that* $B_n(q)$ *is rank-unimodal and Sperner.*

Exercise 4.5. *(1). If* $S_1 = S_2 = ... = S_k$*, then* P *is clear rank-unimodal.*

(2). If $S_i \cap S_j = \emptyset$*,* $i \neq j$*, i.e,* $S_1, S_2, ..., S_k$ *does not intersect with each other. Let* $\#S_1 = \#S_2 = ... = \#S_k = n$*. By* $\binom{n}{0}, \binom{n}{1}, ..., \binom{n}{n}$ *is the n-th line of the Pascal triangle. This is rank-unimodal. So* $k\binom{n}{0}, k\binom{n}{1}, ..., k\binom{n}{n}$ *is clear rank-unimodal.*

....

Exercise 4.6. *Let* $C_1, C_2, ..., C_N$ *be a symmetric chain decomposition,* $A_1, A_2, ...A_m$ *be all antichains of* P*. Suppose* $P = P_1 \uplus P_2 \uplus \cdots \uplus P_n$*, where* $P_{i+1}, P_{i+2}, ..., P_{i+j}$ *is the largest size* j *levels of* P*. Suppose* A *be set of the largest size of a union of* j *antichains. Now, we prove*

$$\#(P_{i+1} \uplus P_{i+2} \uplus \cdots \uplus P_{i+j}) = \#A.$$

First, the $P' = P_{i+1} \uplus P_{i+2} \uplus \cdots \uplus P_{i+j}$ *is a union of* j *antichains, So we have* $\#P' \leq \#A$*. Next, we prove* $\#P' \geq \#A$*. By*

$$\#A = \sum_{k=1}^{N} \#(A \cap C_k), \quad \#P' = \sum_{k=1}^{N} \#(P' \cap C_k),$$

we just need to prove $\#(A \cap C_k) \leq \#(P' \cap C_k)$*. Because* A *is a union of* j *antichains, we know* $\#(A \cap C_k) \leq j$ *(Pigeonhole Principle). By* P' *is a union of* j *levels of* P*, we have* $\#(P' \cap C_k) \leq j$*. If* $\#(P' \cap C_k) = j$*, then we have* $\#(A \cap C_k) \leq \#(P' \cap C_k)$*. If* $\#(P' \cap C_k) < j$*, then* $C_k \subseteq P'$*. (if not, there is* $x \in C_k \setminus P'$*. Then there is* $x' \in C_k$*. So* $\#(P' \cap C_k) = j$*.) So* $P' \cap C_k = C_k$ *and* $A \cap C_k \subseteq C_k = P' \cap C_k$*. Therefore* $\#(A \cap C_k) \leq \#(P' \cap C_k)$*. This completes the proof.*

Exercise 4.7. (a)*. Obviously.*

(b)*. This is Dilworth's theorem. See [1].*

(R. P. Dilworth, A decomposition theorem for partially ordered sets, Annals of Mathematics, 51 (1950), 161–166.)

5 Group Actions on Boolean Algebras

Exercise 5.1. (a). *The ι is identity element and $\pi = (12)$. Note the subset $\{i,j\}$ of $\{1,2,3,4\}$ denote ij. By $\pi\cdot 1 = 2, \pi\cdot 2 = 1, \pi\cdot 3 = 3, \pi\cdot 4 = 4$, we have*

$$\pi\cdot 12 = 12,\ \pi\cdot 13 = 23,\ \pi\cdot 14 = 24,\ \pi\cdot 23 = 13,$$
$$\pi\cdot 24 = 14,\ \pi\cdot 34 = 34,\ \pi\cdot 123 = 123,\ \pi\cdot 124 = 124,$$
$$\pi\cdot 234 = 134,\ \pi\cdot 134 = 234,\ \pi\cdot 1234 = 1234,$$

Each point (orbit) in the following figure (Figure 19) is marked with an element in the orbit:

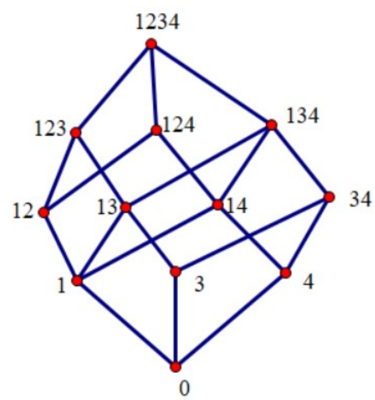

Figure 19:

(b). *By the orbit of $\pi = (12)(34)$:*

$$\{1,2\},\ \{3,4\},\ \{12\},$$
$$\{13,24\},\ \{14,23\},\ \{34\},$$
$$\{123,124\},\ \{134,234\},\ \{1234\},$$

The Hasse diagram as follow (Figure 20):

Exercise 5.2. *The Hasse diagram as follow:*

The size of the largest antichain is 6. There are $2^4 - 1 + 2 + 3 = 20$ such antichains.

Exercise 5.3. *The Partial order relationships can be uniquely determined by Hasse diagram. The example as follow: Let $G = \{e, \pi\}$ and $\pi : 6 \to 7; 7 \to 6; 4 \to 5; 5 \to 4$. Other unchanged. So $\pi^2 = e$ and G is a group. Therefore the Hasse diagram of P/G as follow: However, the largest antichain is $\{2,3,8,9\}$. This are 4 elements. So it is not Sperner.*

Exercise 5.4. .

Exercise 5.5. (a). *In the poset $B_n(q)$ (graded of rank n), let $x \in \{i_1, i_2, ..., i_k\} \in B_n, 1 \leqslant k \leqslant n$. Assuming that B_n is equivalent to the linear arrangement of the set $\{0,1\}$ with a length*

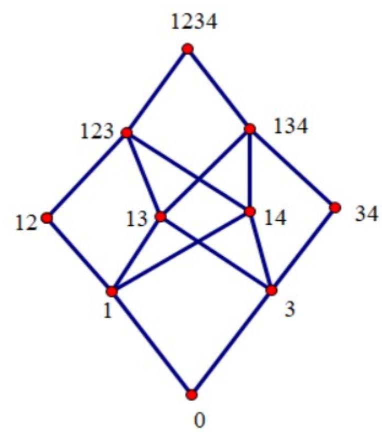

Figure 20:

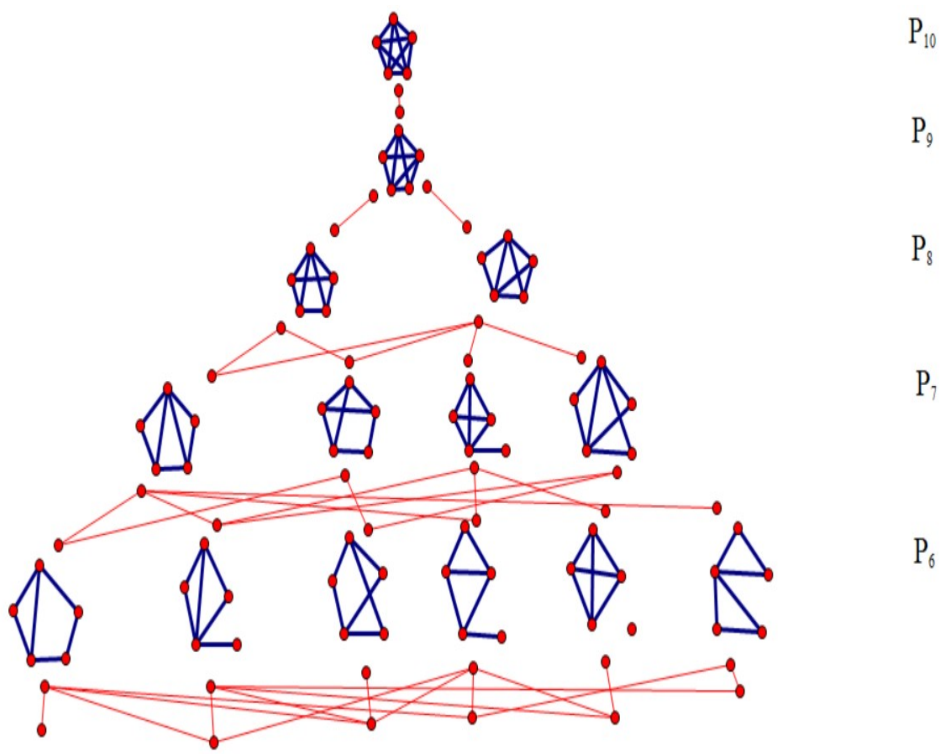

Figure 21:

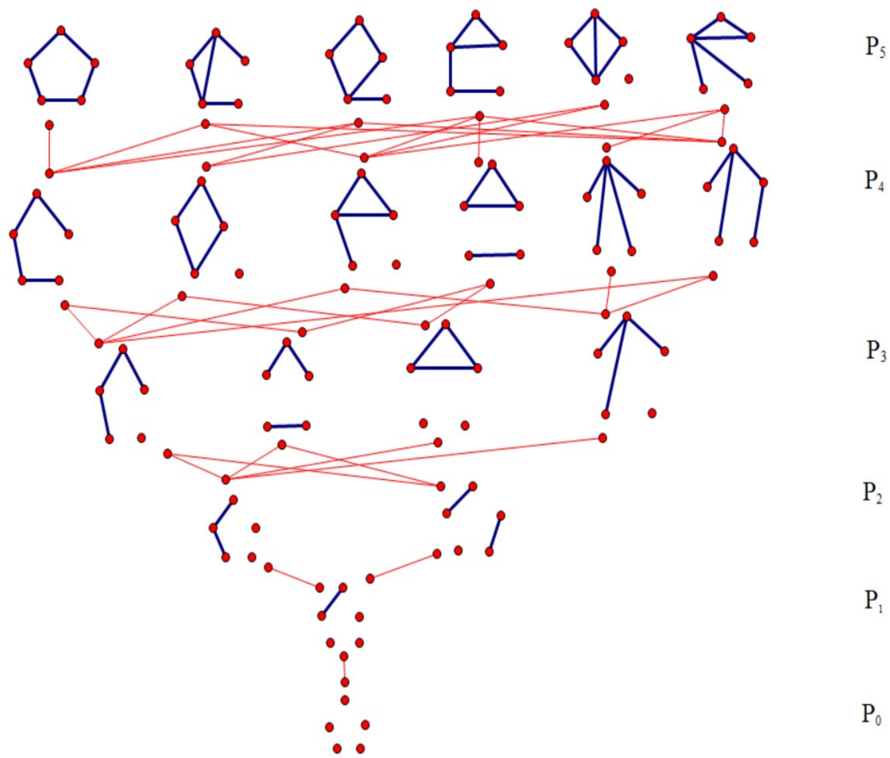

Figure 22:

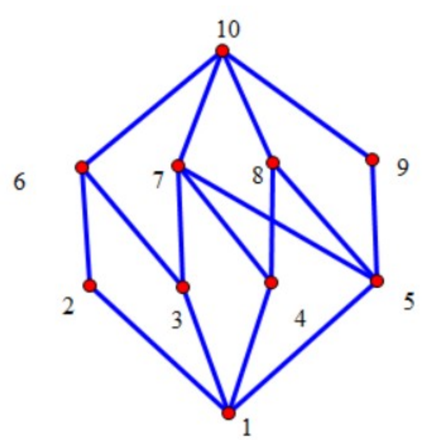

Figure 23:

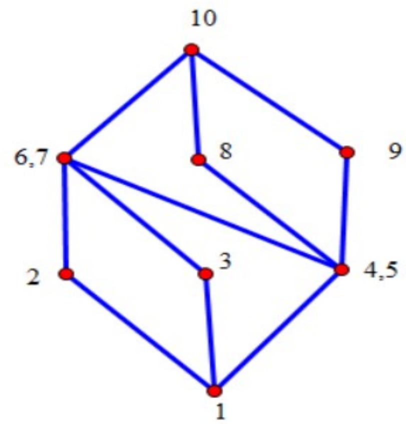

Figure 24:

of n. If $x \in \{i_1, i_2, ..., i_k\} \in B_n$, then we have the arrangement $(0...1...1...1...0)$, where the positions of 1 appears are $i_1, i_2, ..., i_k$, respectively. Obviously, this is a bijection.

Let $G =< (123...n) >$, for any $\pi \in G$, we have $\pi \cdot x = \{\pi(i_1), \pi(i_2), ..., \pi(i_k)\} \in B_n$. This group action corresponds to a circular arrangement of all $\{0,1\}$ sets of length n. By the N_n denote the set of all $(0,1)-$necklaces of length n, we know $N_n \cong B_n/G$. Therefore N_n is rank-symmetric, rank-unimodal and Sperner.

Note: The proof of rank-symmetric. Suppose $x \in N_n$, the x denote the $(0,1)-$necklaces of length n. We can obtain $\overline{x}$ from by changing 0 to 1 and 1 to 0. So $\overline{x} \in N_n$. If x_i is a necklaces of rank i, then $\overline{x}$ is a necklaces of rank $n-i$ in N_n. (exist $x_i \Leftrightarrow$ exist $\overline{x}$). Let $|x_i|$ denote the weight of x_i. We have $|x_i| = i \Leftrightarrow |\overline{x}| = n-i$. So $|(N_n)_i| = |(N_n)_{n-i}|$ and N_n is rank-symmetric.

(b). ......

Exercise 5.6. .

Exercise 5.7. *We have $M = \{a_1 \cdot 1, a_2 \cdot 2, ..., a_k \cdot k\}$. Let $n = \sum_{i=1}^{k} a_i$ and B_n is Boolean algebra of rank n. Let's change the expression for the set A:*

$$M = \{1^1, 1^2, ..., 1^{a_1}, 2^{a_1+1}, ..., 2^{a_1+a_2}, 3^{a_1+a_2+1}, ..., k^{\sum_{i=1}^{k-1} a_i+1}, ..., k^{\sum_{i=1}^{k} a_i}\}.$$

Obviously, this set has a_i i's. We need to find a suitable n and a subgroup G of G_n such that $B_M \cong B_n/G$. Due to the fact that the element in the first layer of B_M is $1, 2, ..., k$, there are k orbits in the first layer of B_n/G. Let

$$G = S_{[1,a_1]} \times S_{[a_1+1,a_1+a_2]} \times \cdots \times S_{[a_1+\cdots a_{k-1}+1,a_1+\cdots+a_k]},\ n = \sum_{i=1}^{k} a_i.$$

So $B_M \cong B_n/G$. By Theorem 5.8, the B_M is rank-symmetric, rank-unimodal and Sperner.

***********************There are errors below**************************

For example, the case $a_1 = 2, a_2 = 3$. We have $M = \{2 \cdot 1, 3 \cdot 2\} = \{1^1, 1^2, 2^3, 2^4, 2^5\}$. Let $G =< (12)(345) >, \pi = (12)(345)$. We have $\pi \cdot 1 = 2, \pi \cdot 2 = 1, \pi \cdot 3 = 4, \pi \cdot 4 = 5, \pi \cdot 5 = 3$. So we have 2 orbits in the first layer: $\{1, 2\}$,$\{3, 4, 5\}$.

Now, we consider the second layer (We use $\rightarrow$ to represent the change under the action of π).

$$\begin{cases} \pi \cdot 12 = 12 \\ \pi \cdot 13 = 24 \rightarrow 15 \rightarrow 23 \rightarrow 14 \rightarrow 25 \rightarrow 13 \\ \pi \cdot 34 = 45 \rightarrow 35 \rightarrow 34 \end{cases}$$

Therefore, there are 3 orbits in the second layer of B_n/G: $\{12\}, \{13, 24, 15, 23, 14, 25\}, \{34, 45, 35\}$.

For the third layer, there are 3 orbits.

$$\begin{cases} \pi \cdot 123 = 124 \rightarrow 125 \rightarrow 123 \\ \pi \cdot 134 = 245 \rightarrow 135 \rightarrow 234 \rightarrow 145 \rightarrow 235 \rightarrow 134 \\ \pi \cdot 345 = 345 \end{cases}$$

For the fourth layer, there are 2 orbits.

$$\begin{cases} \pi \cdot 1234 = 1245 \rightarrow 1235 \rightarrow 1234 \\ \pi \cdot 1345 = 2345 \end{cases}$$

Obviously, the fifth layer is $\{12345\}$. Now, the Hasse diagram as follow (in Figure 25):

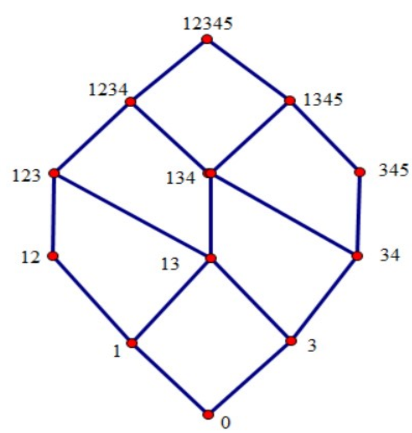

Figure 25:

Exercise 5.8. .

Exercise 5.9. *(a).......*

(b). By (a), if $p \neq 0(mod 4)$, then φ is invertible, i.e, φ is a bijection. For different graph G and G', we have $\varphi(G) = G_1 + G_2 + ... + G_p$ and $\varphi(G') = G'_1 + G'_2 + ... + G'_p$, i.e., $\varphi(G) \neq \varphi(G')$. So $G_1, G_2, ..., G_p$ uniquely determine G.

(c).....

(d). Two four vertex graphs with different edge numbers (in Figure 26).

Figure 26:

Then the unlabeled graph of G *is* G_i *(in Figure 27):*

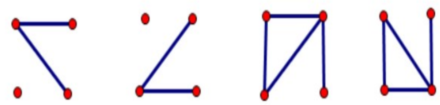

Figure 27:

Then the unlabeled graph of G' *is* G_i *(in Figure 28).*

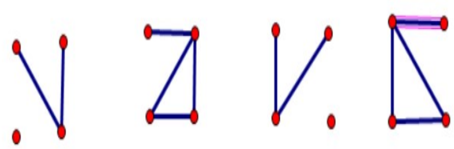

Figure 28:

Therefore, G *and* G' *have the same unlabeled graph* G'

(e). *If* $p = 4$, G *is not necessarily weakly switching-reconstructible. The counterexample is as follows (in Figure 29):*

They have the same label graph G_i *(in Figure 30)*

Exercise 5.10. *If* $i = j$, *The conclusion is clearly hold.*

If $0 \leq i < j \leq \frac{n}{2}$, *we use induction: If* $i = j - 1$, *then* $|S \cup T| \leq 2j - 2 \leq n - 2$. *So there exist* $x \in X \setminus (S \cup T)$, *such that* $S \cup \{x\}, T \cup \{x\}$ *is the* $j-$*element subsets. Therefore, there exist* $\pi \in G$ *such that* $\pi(S \cup \{x\}) = T \cup \{x\}$. *So we have* $p_j = 1$ *in* B_x/G.

The B_x/G *is a poset graded of rank* n, *rank-symmetric, rank-unimodal and Sperner. So we have*

$$1 \leq p_0 \leq p_1 \leq p_2 \leq \cdots \leq p_j = 1,$$

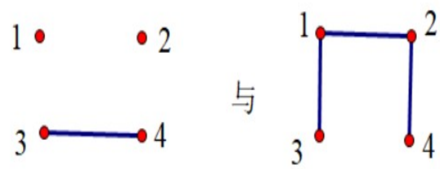

Figure 29:

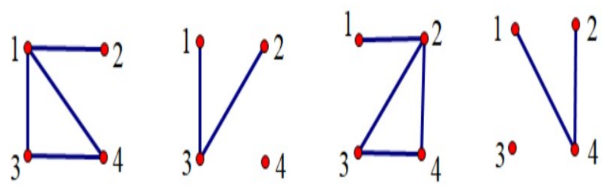

Figure 30:

Then

$$p_0 = p_1 = p_2 = \cdots = p_j = 1.$$

Therefore G acts transitively on $i-$element subsets for all $0 \leq i \leq j$.

Exercise 5.11. *In Figure 31:*

By $G = <(12345)>$, we have

$$\begin{cases} G \cdot 1 = \{1, 2, 3, 4, 5\} \\ G \cdot 12 = \{12, 23, 34, 45, 15\} \\ G \cdot 13 = \{13, 24, 35, 14, 25\} \\ G \cdot 123 = \{123, 234, 345, 145, 125\} \\ G \cdot 124 = \{124, 235, 134, 245, 135\} \\ G \cdot 1234 = \{1234, 2345, 1345, 1245, 1235\} \\ G \cdot 12345 = \{12345\}. \end{cases}$$

We need to calculate

$$\begin{cases} \widehat{U_2}(12) = c_1 \cdot 123 + c_2 \cdot 124; \\ \widehat{U_2}(13) = c_1' \cdot 123 + c_2' \cdot 124. \end{cases}$$

By

$$\begin{aligned} U_2(v_{12}) = c_1 \cdot v_{123} + c_2 \cdot v_{124} = U_2\left(\sum_{x \in 12} x\right) \\ &= U_2(12 + 23 + 34 + 45 + 15) \\ &= 2(123 + 125 + 145 + 234 + 345) + (124 + 235 + 134 + 245 + 135) \end{aligned}$$

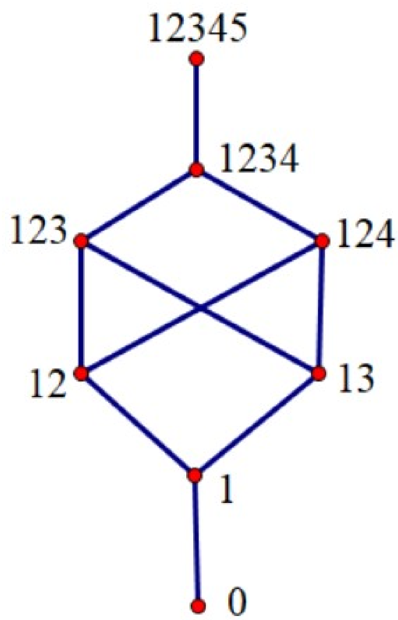

Figure 31:

$$= 2 \cdot v_{123} + 1 \cdot v_{124},$$

we have $\widehat{U_2}(12) = 2 \cdot 123 + 1 \cdot 124$.

Similarly, by

$$\begin{aligned}
U_2(v_{13}) &= c_1' \cdot v_{123} + c_2' \cdot v_{124} = U_2\left(\sum_{x \in 13} x\right) \\
&= U_2(13 + 24 + 35 + 14 + 15) \\
&= 2(124 + 235 + 134 + 245 + 135) + (123 + 234 + 345 + 145 + 125) \\
&= 1 \cdot v_{123} + 2 \cdot v_{124},
\end{aligned}$$

we have $\widehat{U_2}(13) = 1 \cdot 123 + 2 \cdot 124$. *So*

$$\widehat{U_2}(12, 13) = (123, 124) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Exercise 5.12. .

Exercise 5.13. .

6 Young Diagrams and q-Binomial Coefficients

Exercise 6.1. .....

Exercise 6.2. *(a)...*

(b): This result was first presented in the article: Germain. Kreweras, "Sur une classe de problèmes de dénombrement liés au treillis des partitions des entiers". Cahiers du Bureau universitaire de recherche opérationnelle Série Recherche, Volume 6 (1965), pp. 9-107.

It is also a special case of the result found on page 242 of the second volume of the book by P. A. MacMahon, Combinatory Analysis (1915,1916), reprinted by Chelsea in 1960.

The authors recommend consulting the paper by G. Kreweras and H. Niederhausen, "Solution of an enumerative problem connected with lattice paths", European Journal of Combinatorics, Volume 2 (1981), pp. 55-60.

See also the sequence [A111910] on the OEIS website.

Exercise 6.3. .....

Exercise 6.4. (a). *By*

					...		
					...		

Let an order matching $\mu : L(2,n)_i \to L(2,n)_{i+1}$ *,* $i < n$ *be* $\mu(\lambda) = \lambda'$. *Let* $\lambda = (x, y)$. *So* $\lambda' = \mu(\lambda) = (x+1, y)$, *where* x *represents the size of the first component,* y *represents the size of the second component. By* $i < n$, *we have* $x+y < n$ *,* $x+1+y < n+1 < 2n$. *This definition is well-define. By the Young Diagrams, we have* $\lambda' \in L(2,n)_{i+1}$ *and* μ *is* $L(2,n)_i \to L(2,n)_{i+1}$ *an order matching.*

$(b), (c)$. *See [4]. Guoce Xin, and Yueming Zhong, an explicit order matching for* $L(3,n)$ *from several approaches and its extension for* $L(4,n)$*.arXiv:2104. 11003v1, and its references.*

(d).....

Exercise 6.5. .....

Exercise 6.6. ....

7 Enumeration Under Group Action

Exercise 7.1. *If we are allowed to flip necklaces over, not just rotate them, then the group becomes the dihedral group of order* 2ℓ. *If* $\ell = 2m$, *i.e.,* ℓ *is even, then we have two cases to flip necklaces over as Figure 32. So we have the terms* $mz_1^2 z_2^{m-1}$ *(type*$(\pi) = (2, m-1)$*) and* mz_2^m *(type*$(\pi) = (0, m)$*).*

Similarly, if $\ell = 2m+1$, *i.e.,* ℓ *is odd, then have one case to flip necklaces over as Figure 33. So we have the term* $\ell z_1 z_2^m$ *(type*$(\pi) = (1, m)$*). By Theorem 7.5 and Theorem 7.10, if* $\ell = 2m$, *then we have*

$$\frac{1}{2\ell}\left(\sum_{d|\ell} \phi(d) z_d^{\ell/d} + mz_1^2 z_2^{m-1} + mz_2^m\right).$$

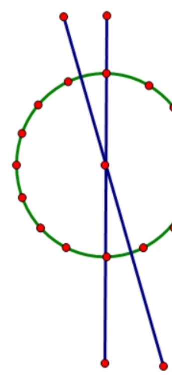

Figure 32:

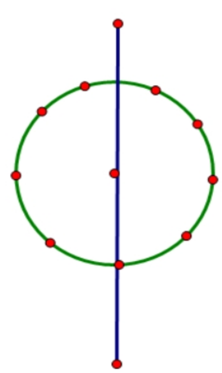

Figure 33:

If $\ell = 2m+1$, then we have

$$\frac{1}{2\ell}\left(\sum_{d|\ell}\phi(d)z_d^{\ell/d} + \ell z_1 z_2^m\right).$$

Exercise 7.2. (*a*). *Mark 1 to 7 from top to bottom for Γ as follows: Due to G being an*

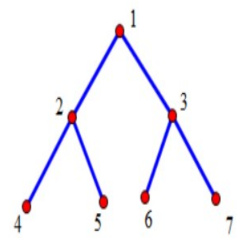

Figure 34:

automorphism group of Γ, we have $e = (1)(2)(3)(4)(5)(6)(7) \in G$ and $(45) \in G, (67) \in G, (45)(67) \in G, (23)(46)(57) \in G, (23)(47)(56) \in G$. Let

$$\begin{array}{ll} \pi_1 = (1)(2)(3)(45)(6)(7) & \pi_2 = (1)(2)(3)(4)(5)(67) \\ \pi_3 = (1)(2)(3)(45)(67) & \pi_4 = (1)(23)(45)(67) \\ \pi_5 = (1)(23)(47)(56). & \end{array}$$

By G is group, we have

$$\pi_6 = \pi_2\pi_4 = (1)(23)(4657) \in G,$$

$$\pi_7 = \pi_1\pi_4 = (1)(23)(4756) \in G.$$

We can verify that $\pi_6.\pi_7$ is indeed an automorphism of Γ. Therefore we have

$$G\{e, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7\}, \#G = 8.$$

So

$$Z_G = \frac{1}{8}(z_1^7 + 2z_1^5 z_2 + z_1^3 z_2^2 + 2z_1 z_2^3 + 2z_1 z_2 z_4).$$

(*b*). *By Theorem 7.5, the number $N_G(n)$ of inequivalent n-colorings of Γ is given by*

$$N_G(n) = \frac{1}{\#G}\sum_{\pi\in G} n^{c(\pi)} = \frac{1}{8}(n^7 + 2n^6 + n^5 + 2n^4 + 2n^3).$$

Alternative approach: by Polya Theorem, we have

$$F_G(r_1, r_2, ..., r_n) = Z_G(r_1 + r_2 + ... + r_n, r_1^2 + r_2^2 + ... + r_n^2, ..., r_1^j + r_2^j + ... + r_n^j, ...).$$

If $r_1 = r_2 = ... = r_n = 1$, then we have

$$N_G(n) = F_G(1, 1, ..., 1) = \frac{1}{8}(n^7 + 2n^6 + n^5 + 2n^4 + 2n^3).$$

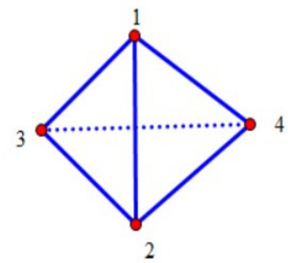

Figure 35:

Exercise 7.3. .

Exercise 7.4. *The four corners of a regular tetrahedron are* $1, 2, 3, 4$ *as Figure 35. Due to the fact that all four triangles of a regular tetrahedron are equilateral triangles, for point 1, we have*

$$|G_1| = \#\{e, (1)(234), (1)(243)\} = 3,$$
$$|O_1| = \{1, 2, 3, 4\} = 4.$$

So we have $|G| = |G_1| \cdot |O_1| = 12$*. In a regular tetrahedron, when considering fixed vertices, it is equivalent to also considering fixed faces. For example, a straight line passing through vertex 1 and the center of a tetrahedron must pass through the center of the base surface. Therefore, When fixing a vertex (face), there are a total of 8 items for circle type* $z_1^1 z_2^1$*, namely:*

$$(1)(234), (1)(243), (2)(134), (2)(143), (3)(142), (3)(124), (4)(123), (4)(132).$$

When fixing the edges, there are 3 *for circle type* z_2^2*, namely:*

$$(12)(34), (13)(24), (14)(23).$$

The circle type of unit element $(1)(2)(3)(4) = e$ *is* z_1^4*.*

(a).

$$Z_G = \frac{1}{12}(z_1^4 + 3z_2^2 + 8z_1 z_3).$$

(b)*. The number of inequivalent* n*-colorings is given by*

$$N_G(n) = \frac{1}{12}(n^4 + 11n^2).$$

(c)*. Now we label the edges of a regular tetrahedron as* $1, 2, 3, 4, 5, 6$ *in Figure 36. When fixing a vertex, there are a total of 8 items for circle type* z_3^2*, namely:*

$$(123)(456), (132)(465), (124)(356), (142)(365),$$

$$(146)(325), (164)(352), (136)(254), (163)(245).$$

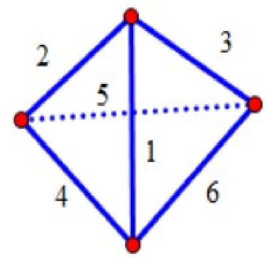

Figure 36:

When fixing the edges, there are 3 for circle type $z_1^2z_2^2$, namely:

$$(2)(6)(34)(15), (3)(4)(15)(26), (1)(5)(34)(26).$$

The circle type of unit element $e = (1)(2)(3)(4)(5)(6)$ is z_1^6. The cycle index polynomial is given by

$$Z_G = \frac{1}{12}(z_1^6 + 8z_3^2 + 3z_1^2z_2^2).$$

The number of inequivalent n-colorings is given by

$$N_G(n) = \frac{1}{12}(n^6 + 8n^2 + 3n^4).$$

Exercise 7.5. .

Exercise 7.6. *By cyclic symmetry, i.e. the rotation of the chain, let $\pi = (123...2n), G = \{1, \pi, \pi^2, ..., \pi^{2n-1}\}$. By Theorem 7.10, we have*

$$\begin{aligned}
F_G(r, b) &= \frac{1}{2n}\sum_{d|2n}\varphi(d)(r^d + b^d)^{\frac{2n}{d}} \\
&= \sum_{i_1, i_2}\kappa(r_1, r_2)r^{i_1}b^{i_2} \\
&= \frac{1}{2n}\sum_{d|2n}\varphi(d)\sum_{i=0}^{\frac{2n}{d}}\binom{\frac{2n}{d}}{i}(r^d)^i(b^d)^{\frac{2n}{d}-i}.
\end{aligned}$$

We want to know the coefficient $\kappa(n, n)$ of r^nb^n. In other words, when $i = \frac{n}{d}$, we want to know the coefficient before

$$r^{d\cdot\frac{n}{d}} \cdot b^{\frac{2n}{d}-\frac{n}{d}} \cdot d = r^nb^n.$$

If $d|n$, i.e., i is an integer, we have

$$\kappa(n, n) = \frac{1}{2n}\sum_{d|n}\varphi(d)\binom{\frac{2n}{d}}{\frac{n}{d}},$$

i.e.,

$$\kappa(n, n) = \frac{1}{2n}\sum_{d|n}\varphi(d)\binom{\frac{2n}{d}}{\frac{n}{d}}.$$

Exercise 7.7. *The M´obius function is given by*

$$\mu(n)=\begin{cases}0 & n=1;\\ (-1)^k & n=p_1p_2...p_k\ (p_i\neq p_j);\\ 0 & otherwise,\end{cases}$$

where p_i is prime number. We have

$$\sum_{d|l} dM_d(n)=n^l.$$

According to the Mobius inversion formula:

$$f(l)=\sum_{d|l} g(d)\Longleftrightarrow g(l)=\sum_{d|l}\mu(d)f(\frac{l}{d}),$$

we have

$$l\cdot M_l(n)=\sum_{d|l}\mu(d)n^{\frac{l}{d}},$$

$$l\cdot M_l(n)=\sum_{d|l}\mu(\frac{l}{d})n^{d},$$

$$M_l(n)=\frac{1}{l}\sum_{d|l}\mu(\frac{l}{d})n^{d}.$$

Exercise 7.8. (a). *We label these ten balls from top to bottom as* $1,2,3,4,5,6,7,8,9,10$ *in Figure 37. Due to the fact that the triangular queue can rotate freely in a two-dimensional*

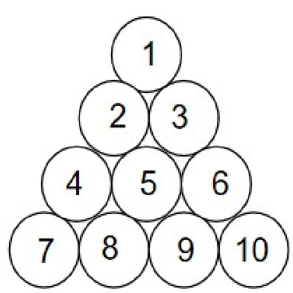

Figure 37:

plane, we have

$$G=\{e,(1710)(286)(349)(5),(1107)(268)(394)(5)\}.$$

The cycle index polynomial is given by

$$Z_G=\frac{1}{3}(z_1^{10}+2z_1z_3^3).$$

Therefore the generating function of inequivalent colorings using the ten colors is given by

$$\begin{aligned}
F_G(r_1, r_2, ..., r_{10}) &= \sum_{i_1, i_2, ..., i_{10}} \kappa(i_1, i_2, ..., i_{10}) r_1^{i_1} r_2^{i_2} ... r_{10}^{i_{10}} \\
&= Z_G(r_1 + r_2 + ... + r_{10}, r_1^2 + r_2^2 + ... + r_{10}^2, ...) \\
&= \frac{1}{3}\left(\left(\sum_{i=1}^{10} r_i\right)^{10} + 2\left(\sum_{i=1}^{10} r_i\right)\left(\sum_{i=1}^{10} r_i^3\right)^3\right).
\end{aligned}$$

(b). By (a), we have

$$F_G(r_1, r_2, ..., r_{10}) = \frac{1}{3}\left(\left(\sum_{i=1}^{10} r_i\right)^{10} + 2\left(\sum_{i=1}^{10} r_i\right)\left(\sum_{i=1}^{10} r_i^3\right)^3\right).$$

The polynomial theorem is

$$(x_1 + x_2 + ... + x_t)^n = \sum \binom{n}{n_1, ..., n_t} x_1^{n_1} x_2^{n_2} ... x_t^{n_t}.$$

Find the inequivalent colorings numbers of 4 red balls, 3 green balls, and 3 chartreuse balls, that is, find the coefficient $\kappa(4, 3, 3, 0, ..., 0)$ of $r_1^4 r_2^3 r_3^3$. We have

$$\begin{aligned}
\kappa(4, 3, 3, 0, ..., 0) &= \frac{1}{3}\left(\binom{10}{4, 3, 3} + 2\binom{3}{1, 1, 1}\right) \\
&= \frac{1}{3}\left(\frac{10!}{4!3!3!} + 2 \cdot 3!\right) \\
&= 1404.
\end{aligned}$$

Find the inequivalent colorings numbers of 4 red balls, 4 turquoise balls, and 2 aquamarine balls, that is, find the coefficient $\kappa(4, 4, 2, 0, ..., 0)$ of $r_1^4 r_2^4 r_3^2$. We have

$$\kappa(4, 4, 2, 0, ..., 0) = \frac{1}{3}\left(\binom{10}{4, 4, 2} + 0\right) = 1050.$$

Exercise 7.9. *Up to the action of D_4, consider $C = \{r, b, y\}$. We calculate the $N_{D_4}(3)$ The elements colored with r are $X \setminus (S \cup T)$. The elements colored with b are: set S. The elements colored with y are: $T \setminus S$. Now, we have*

$$Z_{D_4} = \frac{1}{8}(2z_2^{32} + 2z_1^8 z_2^{28} + 2z_4^{16} + z_2^{32} + z^{64}).$$

Therefor, let $z_i = 3$, we obtain

$$N_{D_4}(3) = \frac{1}{8}(3 \cdot 3^{32} + 2 \cdot 3^{36} + 2 \cdot 3^{16} + 3^{64}).$$

Exercise 7.10. *By $f(n)$ denote the number of inequivalent n-colorings of X, we have*

$$f(n) = \frac{1}{\#G} \cdot \sum_{\pi \in G} n^{c(\pi)},$$

where $c(\pi)$ is the number of cycles in π. Furthermore, we have

$$f(n) = \frac{1}{\#G}(c_1 n + c_2 n^2 + ... + c_i n^i + ... + c_l n^l),$$

where c_i is the number of permutations with i cycles in G. By $\#X = l$, then the permutation corresponding to $c_l = 1$ is $(1)(2)...(l)$. We have

$$\lim_{n\to\infty} \frac{f(n)}{n^l} = \lim_{n\to\infty} \frac{c_1 n + c_2 n^2 + ... + c_{l-1} n^{l-1} + n^l}{\#G \cdot n^l} = \frac{1}{\#G}.$$

Exercise 7.11. *First, we have*

$$\begin{aligned} f(n) &= \frac{1}{\#G} \cdot \sum_{\pi \in G} n^{c(\pi)} \\ &= \frac{1}{\#G}(c_1 n + c_2 n^2 + ... + c_{l-1} n^{l-1} + n^l) \\ &= \frac{1}{443520}(n^{11} + 540 n^9 + ... + 10n). \end{aligned}$$

where c_i is the number of permutations with i cycles in G. Two polynomials that are equal have the same coefficients.

(a). *The order of G is $\# = 443520$*

(b). *There is an identity element $e = (1)(2)...(l)$ (z_1^l) in any group G. So we have $l = 11$, i.e., $\#X = 11$.*

(c). *Let π be the transpositions in G. Then the cycle index polynomial of π is $z_\pi = z_1^9 z_2$. The coefficient of n^{10} in $f(n)$ is 0. So the number of transpositions in G is 0.*

(d). *By Burnside's lemma, the number of orbits of G (action on X) is*

$$|X/G| = \frac{1}{\#G} \sum_{\pi \in G} \#Fix(\pi),$$

$$Fix(\pi) = \{y \in x, \pi(y) = y\},$$

where $\#Fix(\pi)$ is the number of cycles of length one in the permutation π.

By there exist the term $10n$ in $f(n)$, and $\#X = 11$, we know that there exist 11-cycles in G. Let $\pi \in G$ be a 11-cycles in G. For any $x \in X$, we have

$$\{\pi^k(x) | k = 1, 2, ..., 11\} = X.$$

So there is 1 orbit.

(e). *We only need to find the non trivial normal subgroup of* G. *By there exist the term* $10n$ *in* $f(n)$, *and* $\#X = 11$, *we know that there exist* 11-*cycles in* G. *Let* $\pi \in G$ *be a* 11-*cycles in* G. *Therefor,* $\langle \pi \rangle$ *is a normal subgroup. Since for any* $\sigma \in G$, *we have*

$$\sigma(i_1 i_2 \cdots i_{11})\sigma^{-1} = (\sigma(i_1)\sigma(i_2)\cdots\sigma(i_{11})) \in \langle \pi \rangle.$$

*********************There are errors below************************

The coefficient of n^9 *in* $f(n)$ *is* 540. *The* n^9 *come form the type* $1^7 2^2$ *and* $1^8 3^1$. *Suppose*

$$\pi = (a_1)(a_2)...(a_7)(a_8a_9)(a_{10}a_{11}), \sigma = (a_1)(a_2)...(a_8)(a_9a_{10}a_{11}),$$

where $a_i \in \{1, 2, ..., 11\}$, $a_i \neq a_j, (i \neq j)$. *If there are* π *types among* 540, *then we have* $H =< \pi >= \{e, \pi\}$. *If there are* σ *types, then we have* $H =< \sigma >= \{e, \sigma, \sigma^2\}$. *The* H *is a non trivial subgroup of* G. *So* G *is not a simple group.*

Exercise 7.12. .

Exercise 7.13. .

Exercise 7.14. .

Exercise 7.15. (a). *We have* $e_6(n) = \#\{\pi | \pi^6 = l\}$. *The factors of* 6 *include* $1, 2, 3, 6$. *By*

$$Z_G = \frac{1}{\#G} \sum_{\pi \in G} z_\pi,$$

and

$$z_\pi = z_1^{c_1} z_2^{c_2} ... z_n^{c_n}, \quad \sum_i i c_i = n.$$

We have $e_6(n) = n! Z_G(z_1 = z_2 = z_3 = z_6 = 1, z_4 = z_5 = 0)$. *By Theorem 7.13, the generating function of* Z_G *is given by*

$$\sum_{l \geqslant 0} Z_G(z_1, z_2, ...)x^l = \exp\left(z_1 x + z_2 \frac{x^2}{2} + z_3 \frac{x^3}{3} + ...\right).$$

We have

$$\begin{aligned}\sum_{n \geqslant 0} e_6(n) \frac{x^n}{n!} &= \exp\left(z_1 x + z_2 \frac{x^2}{2} + z_3 \frac{x^3}{3} + z_6 \frac{x^6}{6}\right)\Big|_{z_1 = z_2 = z_3 = z_6 = 1} \\ &= \exp\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^6}{6}\right).\end{aligned}$$

(b). *For any* $k \geqslant 1$, *by* $e_k(n) = \#\{\pi \in G_n | \pi^k = l\}$, *we similar have*

$$\sum_{n \geqslant 0} e_k(n) \frac{x^n}{n!} = \exp\left(\sum_{m|k} \frac{x^m}{m}\right).$$

Exercise 7.16. *(a). Let $f(n)$ be the number of permutation in the symmetric group G_n all of whose cycles have even length. So we have*

$$f(n) = n! Z_{G_n}(i \ \ is \ \ even, \ z_i = 1; \ i \ \ is \ \ odd, \ \ z_i = 0),$$

where

$$Z_G = \frac{1}{|G|} \sum_{\pi \in G} z_\pi,$$

and

$$z_\pi = z_1^{c_1} z_2^{c_2} ... z_n^{c_n}, \sum_i i c_i = n.$$

By the generating function is (Theorem 7.13)

$$\sum_{l \geqslant 0} Z_{G_n}(z_1, z_2, ...) x^l = \exp\left(z_1 x + z_2 \frac{x^2}{2} + z_3 \frac{x^3}{3} + ... \right).$$

we have

$$\sum_{n \geqslant 0} \frac{f(n)}{n!} x^n = \exp\left(\frac{1}{2} x^2 + \frac{1}{4} x^4 + \frac{1}{6} x^6 + ... \right).$$

By

$$\ln(1+x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + ...,$$

and

$$-\ln(1-x) = x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + ...,$$

we have

$$-\frac{1}{2}(\ln(1+x) + \ln(1-x)) = \frac{1}{2} x^2 + \frac{1}{4} x^4 + \frac{1}{6} x^6 + ...$$

Therefore we have

$$\sum_{n \geqslant 0} \frac{f(n)}{n!} x^n = \exp\left(-\frac{1}{2} \ln(1-x^2) \right) = \left(1-x^2\right)^{-\frac{1}{2}}.$$

(b). By (a), we have

$$\sum_{n \geqslant 0} \frac{f(n)}{n!} x^n = (1-x^2)^{-\frac{1}{2}} = \sum_{n \geqslant 0} \binom{-\frac{1}{2}}{n} (-x^2)^n = \sum_{n \geqslant 0} (-1)^n \binom{-\frac{1}{2}}{n} x^{2n}.$$

When n is odd, the coefficient of x^n is 0. We have $f(n) = 0$. When n is even, we have

$$\begin{aligned} \sum_{n \geqslant 0} \frac{f(n)}{n!} x^n &= \sum_{m \geqslant 0} (-1)^m \binom{-\frac{1}{2}}{m} x^{2m} \\ &= \sum_{m \geqslant 0} \frac{(-1)^m (-1)^m \cdot 1 \cdot 3 ... (2m-1)}{m! \cdot 2^m} x^{2m} \end{aligned}$$

$$= \sum_{m \geqslant 0} \frac{1^2 \cdot 3^2 ... (2m-1)^2}{(2m)!} x^{2m}.$$

Let $n = 2m$*, then we have* $f(n) = 1^2 \cdot 3^2 \cdot ... \cdot (2m-1)^2 = 1^2 \cdot 3^2 \cdot ... \cdot (n-1)^2$*. Therefore we have*

$$f(n) = \begin{cases} 0 & n \ \ is \ \ odd, \\ 1^2 \cdot 3^2 \cdot ... \cdot (n-1)^2 & n \ \ is \ \ even. \end{cases}$$

(c) See Figure 38.

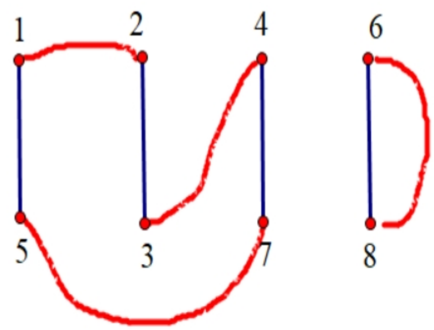

Figure 38:

If there is no square in the above equation, consider a perfect match (blue line segment), where 1 and 5 form a circle, 2 and 3 form a circle, and so on. There are $2n-1$ *paired with 1,* $2n-3$ *paired with 2, and so on. If there is square, matching again (blue line segment + red line segment) yields* $(157432)(68)$*. This operation can return, which is a bijective matching.*

Exercise 7.17. .

Exercise 7.18. *The number of inequivalent* m*-colorings of* X *(about* G*) is*

$$N_G(m) = \frac{1}{\#G} \sum_{\pi \in G} m^{c(\pi)}, \quad \#G = n!.$$

We have

$$N'_G(m) = \frac{1}{n!} \sum_{\pi \in G} c(\pi) m^{c(\pi)-1},$$

and

$$N''_G(m) = \frac{1}{n!} \sum_{\pi \in G} c(\pi)(c(\pi)-1) m^{c(\pi)-2}.$$

If $m = 1$*, we have*

$$N''_G(1) = \frac{1}{n!} \sum_{\pi \in G} c(\pi)(c(\pi)-1) = f(n).$$

By Polya's theorem, we have

$$F_G(r_1, r_2, ...) = Z_G(r_1 + r_2 + ..., r_1^2 + r_2^2 + ..., ..., r_1^j + r_2^j + ...),$$

and

$$N_G(m) = F_G(1,1,1,...) = Z_G(m,m,m,...).$$

By Theorem 7.13, we have

$$\begin{aligned}\sum_{n\geqslant 0} N_G(m)t^n &= \sum_{n\geqslant 0} Z_G(m,m,m...)t^n \\ &= \exp\left(mt + m\frac{t^2}{2} + m\frac{t^3}{3} + ...\right) \\ &= \exp\left(m\left(t + \frac{t^2}{2} + \frac{t^3}{3} + ...\right)\right).\end{aligned}$$

Taking the derivative of m on both sides yields:

$$\sum_{n\geqslant 0} N'_G(m)t^n = \exp\left(m\left(t + \frac{t^2}{2} + \frac{t^3}{3} + ...\right)\right)\cdot\left(t + \frac{t^2}{2} + \frac{t^3}{3} + ...\right),$$

and

$$\sum_{n\geqslant 0} N''_G(m)t^n = \exp\left(m\left(t + \frac{t^2}{2} + \frac{t^3}{3} + ...\right)\right)\cdot\left(t + \frac{t^2}{2} + \frac{t^3}{3} + ...\right)^2$$

By

$$\sum_{n\geqslant 0} f(n)t^n = \sum_{n\geqslant 0} N''_G(1)t^n = \exp\left(t + \frac{t^2}{2} + \frac{t^3}{3} + ...\right)\cdot\left(t + \frac{t^2}{2} + \frac{t^3}{3} + ...\right)^2,$$

we have (come form the proof of Theorem 7.13)

$$\exp\left(t + \frac{t^2}{2} + \frac{t^3}{3} + ...\right) = \frac{1}{1-t}.$$

By

$$\int (1 + t + t^2 + ...)dt = \int \frac{1}{1-t}dt,$$

and

$$t + \frac{1}{2}t^2 + \frac{1}{3}t^3 + ... = -\ln(1-t),$$

we have

$$\sum_{n\geqslant 0} f(n)t^n = \frac{1}{1-t}\cdot(-\ln(1-t))^2 = \frac{(\ln(1-t))^2}{1-t}.$$

Exercise 7.19. (a). *Incomplete.*

The quotient poset B_n/G is rank-symmetric, rank-unimodal and a graded poset of rank n. When n is odd, we have $p_i = P_{n-i}, (i = 0,1,2,...,n)$. If i is even, then $n-i$ is odd. If i is odd, then $n-i$ is even. At this time, the number of elements of the ith level equal the number of elements of $n-i$th level. The number of elements of even rank $0,2,4,...,\frac{n-1}{2}$ equal the number of elements of odd rank $n, n-2, n-4, ..., \frac{n+1}{2}$.

When n is even. ....

(b). ....

Exercise 7.20. *The $c(l,k)$ is the number of permutations in G_l with k cycles. The generating function of the sequence $c(l,1), c(l,2), ..., c(l,l)$ is given by (see page 88)*

$$\sum_{k=1}^{l} c(l,k)x^k = x(x+1)(x+2)...(x+l-1).$$

Let $c(l,0) = 0$, we have

$$\sum_{k=0}^{l} c(l,k)x^k = x(x+1)(x+2)...(x+l-1).$$

From the right-hand of the above equation, it can be seen that this equation is a real coefficient polynomial, and its roots are real numbers $0, -1, -2, ..., -l+1$. By Theorem 5.12 (Newton), the sequence $c(l,0), c(l,1), ...c(l,l)$ is strongly log-concave. So the sequence $c(l,1), c(l,2), ...c(l,l)$ is strongly log-concave.

8 A Glimpse of Young Tableaux

Exercise 8.1. *By Figure 39 and hook-length formula*

$$f^\lambda = \frac{6!}{5 \cdot 4 \cdot 2 \cdot 2 \cdot 1 \cdot 1} = 9,$$

there are 9 standard Young tableaux as Figure 40.

5	4	2	1
2	1		

Figure 39:

Exercise 8.2. *We need to proof that the number of SYT of shape $\lambda = (n,n)$ is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. By*

n+1	*n*	*n-1*	...	*3*	*2*
n	*n-1*	*n-2*	...	*2*	*1*

and the hook-length formula, we have

$$f^\lambda = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1}\binom{2n}{n} = C_n.$$

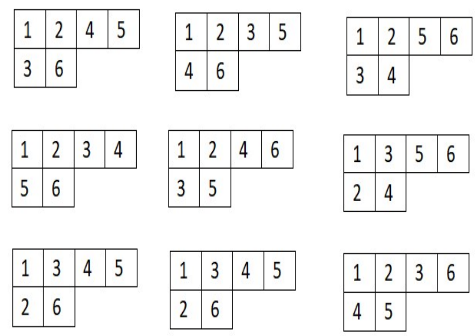

Figure 40:

Exercise 8.3. *The $L(m,n)$ is the set of all partitions with at most m parts and with largest part at most n, i.e., $L(m,n)$ is the set of the Young diagram that is contained in a $m \times n$ rectangle. Its partial order relationship is an inclusion relationship. The $L(m,n)$ is graded poset of rank mn. Therefore, each maximal chain of $L(m,n)$ can be seen as a walk from empty partition $\emptyset$ to $\lambda = nnn...n$ (m $n's$), and the type of the walk is $W = U^{mn}$.*

For $L(4,4)$, the number of maximal chains are in the poset $L(4,4)$ is the value of $\alpha(U^{4\cdot 4}, \lambda)$, where $\lambda = 4444$. By the diagram

7	6	5	4
6	5	4	3
5	4	3	2
4	3	2	1

and the hook-length formula, we have

$$\alpha(U^{16}, \lambda) = f^{\lambda} = \frac{16!}{7 \cdot 6^2 \cdot 5^3 \cdot 4^4 \cdot 3^3 \cdot 2^2} = 24024.$$

Exercise 8.4. *Method 1: The $c(\lambda)$ denote the number of corner squares (distinct parts) of partition λ. So $\sum_{\lambda \vdash n} c(\lambda)$ is the number of edges in graphs $Y_{n-1,n}$. By Corollary 8.10, the number of ways to choose a partition $\lambda \vdash n$, then delete a square form λ, then insert a square, ultimately change back to λ is*

$$\sum_{s=1}^{n} [p(n-s) - p(n-s-1)]s.$$

This number just right is the number of edges in graph $Y_{n-1,n}$. Therefore we have

$$\begin{aligned}\sum_{\lambda \vdash n} c(\lambda) &= \sum_{s=1}^{n} [p(n-s) - p(n-s-1)]s \\ &= [p(n-1) - p(n-2)] \cdot 1 + [p(n-2) - p(n-3)] \cdot 2 + ... \\ &\quad + [p(2) - p(1)] \cdot (n-2) + [p(1) - p(0)] \cdot (n-1) + [p(0) - p(-1)] \cdot n \\ &= p(n-1) + p(n-2) + ... + p(1) + p(0),\end{aligned}$$

where $p(-1) = 0$.

Method 2: (Combinatorial proof) Observe the relationship between the number of corner square in the partitions of 4 *and* $3, 2, 1, \emptyset$ *in Figure 41 : For the correspondence, we delete the*

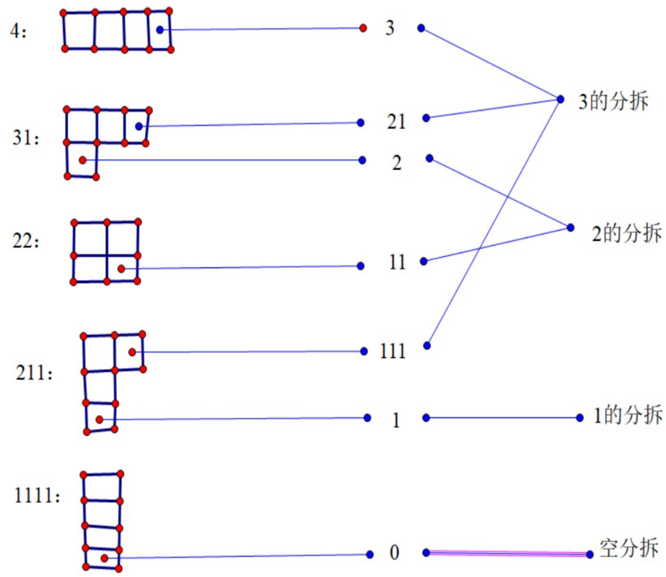

Figure 41:

points and their corresponding columns from the original partition on the left side in Figure 41, resulting in a partition of $3, 2, 1, \emptyset$. *We have*

$$\sum_{\lambda \vdash 4} = p(0) + p(1) + p(2) + p(3).$$

By comparing the partitions λ' *of* $3, 2, 1, \emptyset$ *with* 4, *we can restore the partition* λ *of* 4. *So this is a bijection. Similarly, we can generalize to the case of* n. *We have*

$$\sum_{\lambda \vdash n} c(\lambda) = p(0) + p(1) + ... + p(n-1).$$

Exercise 8.5. *For any partition* λ, *we will draw its Yang diagram. We delate the two connected squares from its Yang diagram, so that the remaining squares are still another Yang diagram of partition. At this time, we have removed one odd length and one even length, and the number of odd and even numbers in the remaining squares remains unchanged. For example, we can see Figure 42. We use* $-, |$ *denote delate square. Recursively, each time the squares are delated in this way, a square of* 1 *is obtained, which is a triangle number, or the square cannot be delated in the ladder diagram shown in Figure 43. (The deletion cannot continue until similar cases occur) At this point, the hook length in each square is odd. Let the number of squares in each row be* $h, h-1, h-2, 2, 1$ *and the numbers in each square are odd. Then the number of odd hook lengths is* $h + (h-1) + (h-2) + ... + 2 + 1 = \frac{h(h+1)}{2}$. *The number of even hook lengths is* 0. *Their difference is a triangular number.*

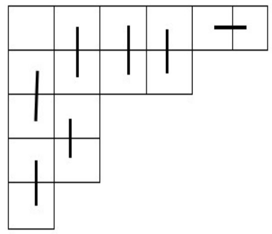

Figure 42:

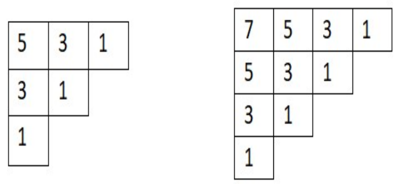

Figure 43:

Exercise 8.6. *See [3, Chapter 7, Exercise 7.16]. R. P. Stanley, Enumerative Combinatorics (volume 2), Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, 1999.*

Exercise 8.7. .

Exercise 8.8. .

Exercise 8.9. .

Exercise 8.10. *Method 1: For the process of adding and delating square, it can be seen as a walk from empty partition $\emptyset$ to $\emptyset$, and the type of the walk is $w = D^{2n}U^nD^nU^{2n}$, $\lambda = \emptyset$. By Theorem 8.4, we have*

$$\begin{aligned}
\alpha(w, \lambda) &= f^\lambda \prod_{i \in S_w} (b_i - a_i) \\
&= 1 \cdot \prod_{i \in S_w} (b_i - a_i) \\
&= 2n(2n-1)...[2n-(n-1)](3n-n)(3n-n-1)...[3n-n-(2n-1)] \\
&= \frac{(2n)!}{n!} \cdot (2n)! = \frac{((2n)!)^2}{n!}.
\end{aligned}$$

Method 2: We consider that action w to 1. The U represents multiplying by x, D represents D_X. We have $D^{2n}U^nD^nU^{2n} \cdot 1$. Therefore we have

$$1 \to x^{2n} \to 2n(2n-1)...(n+1)x^n \to 2n(2n-1)...(n+1)x^{2n} \to 2n(2n-1)...(n+1)[(2n)!].$$

We obtain

$$2n(2n-1)...(n+1)[(2n)!] = \frac{((2n)!)^2}{n}.$$

Exercise 8.11. *See [3, Chapter 7, Exercise 7.15]. R. P. Stanley, Enumerative Combinatorics (volume 2), Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, 1999.*

Exercise 8.12. *By $DU = UD + I$ and $DU^i = U^iD + iU^{i-1}$, we have*

$$\begin{aligned} D^2U^2 &= D(DU^2) = D(U^2D + 2U) = DU^2D + 2DU \\ &= (U^2D + 2U)D + 2(UD + I) \\ &= U^2D^2 + 4UD + 2I, \end{aligned}$$

and

$$\begin{aligned} D^3U^3 &= D^2(DU^3) = D^2(U^3D + 3U^2) \\ &= D(DU^3D + 3DU^2) \\ &= D((U^3D + 3U^2)D + 3(U^2D + 2U)) \\ &= D(U^3D^2 + 6U^2D + 6U) \\ &= (U^3D + 3U^2)D^2 + 6(U^2D + 2U)D + 6(UD + I) \\ &= U^3D^3 + 9U^2D^2 + 18UD + 6I. \end{aligned}$$

Note: For 12 and 14, let's take a different approach: Let $D = \frac{d}{dx} = D_x$ be a derivative operator and $U = x$. So $DU = UD + I$ is equivalent to $D_x \cdot x = xD_x + 1$. Let's apply it to a simple polynomial x^n as follows: We have

$$D^3U^3 \cdot x^n = D_x^3x^3 \cdot x^n = D_x^3 \cdot x^{n+3} = (n+3)(n+2)(n+1) \cdot x^n.$$

Suppose

$$D^3U^3 = c_1U^3D^3 + c_2U^2D^2 + c_3UD + c_4.$$

We consider

$$(c_1U^3D^3 + c_2U^2D^2 + c_3UD + c_4) \cdot x^n = (c_1n(n-1)(n-2) + c_2n(n-1) + c_3n + c_4) \cdot x^n.$$

Let $n = 0$, we obtain $c_4 = 6$. Let $n = 1$, we obtain $c_3 = 18$. Similarly, we obtain $c_2 = 9$, $c_1 = 1$.

Exercise 8.13. .

Exercise 8.14. *We use mathematical induction method. For $n = 1$, $UD = DU$ always holds true. For $n = 2$, we have*

$$U^2D^2 = (UD - I)UD = UDUD - UD = U(UD + I)D - UD = U^2D^2.$$

Assuming the equation holds for $n - 1$, we have

$$U^{n-1}D^{n-1} = (UD - (n-2)I)...(UD - I)UD.$$

Therefor we have

$$\begin{aligned}
U^nD^n &= U(U^{n-1})DD^{n-1} = U(U^{n-1}D)D^{n-1} \\
&= U(DU^{n-1} - (n-1)U^{n-2})D^{n-1} \\
&= UDU^{n-1}D^{n-1} - (n-1)U^{n-1}D^{n-1} \\
&= (UD - (n-1)I)U^{n-1}D^{n-1} \\
&= (UD - (n-1)I)(UD - (n-2)I)...(UD - I)UD.
\end{aligned}$$

This completes the proof.

Exercise 8.15. .

Exercise 8.16. .

Exercise 8.17. *(a). Let* $S = \{1, 2, ..., n\} = S_1 \cup S_2 \cup ... \cup S_t$*. Let* $n \in S_1$, $\#S_1 = k+1$ *for a certain* k, $0 \leq k \leq n-1$*. Therefore,* $S_2, ..., S_t$ *is a partition of an* $[n-1-k]$*-element set. We have*

$$B(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} B(n-1-k) = \sum_{k=1}^{n} \binom{n-1}{k-1} B(n-k).$$

Now we take the derivative of $F(x)$ *and obtain*

$$\begin{aligned}
F'(x) &= \sum_{n\geq 1} B(n) \frac{x^{n-1}}{(n-1)!} \\
&= \sum_{n\geq 1} \sum_{k=0}^{n-1} \binom{n-1}{k} B(n-1-k) \frac{x^{n-1}}{(n-1)!} \\
&= \sum_{n\geq 1} \sum_{k=1}^{n} \binom{n-1}{k-1} B(n-k) \frac{x^{n-1}}{(n-1)!} \\
&= \sum_{n\geq 1} \sum_{k=1}^{n} \frac{1}{(k-1)!(n-k)!} B(n-k) x^{n-1} \\
&= \sum_{n\geq 1} \sum_{k=1}^{n} \frac{x^{k-1}}{(k-1)!} B(n-k) \frac{x^{n-k}}{(n-k)!} \\
&= \sum_{j=0}^{\infty} (\sum_{i=0}^{\infty} \frac{x^j}{j!} B(i) \frac{x^i}{i!}) \\
&= \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{i=0}^{\infty} B(i) \frac{x^i}{i!} \\
&= e^x F(x).
\end{aligned}$$

So we have

$$\frac{dF(x)}{dx} = e^x F(x),$$

i.e.,

$$\frac{dF(x)}{F(x)} = e^x dx.$$

We integrate both sides simultaneously (with initial conditions) to obtain

$$F(x) = e^{e^x - 1}.$$

(b).............

Exercise 8.18. .

Exercise 8.19. *(a). Let r be the number of different components in the partition λ. Then $U(\lambda)$ is a sum of $r+1$ terms and $D(\lambda)$ is a sum of r terms. For any λ, the coefficient of the partition λ in DX is $r+1$. The coefficient of the partition λ in $(U+I)X$ is $r+1$. Therefore we have $DX = (U+I)X$.*

(b). We have $s_0 = 1, s_1 = 1, s_2 = 2, s_3 = 4, s_4 = 10$, and

$$s_n = \sum_{\lambda \vdash n} f^\lambda.$$

(c). We use mathematical induction method. For $n = 0$, $DX = (U+I)X$ always holds true. Assuming the equation holds for n, we have

$$D^n X = (UD^{n-1} + D^{n-1} + (n-1)D^{n-2})X.$$

Therefor we have

$$\begin{aligned} D^{n+1}X &= D(UD^{n-1} + D^{n-1} + (n-1)D^{n-2})X \\ &= DUD^{n-1}X + D^nX + (n-1)D^{n-1}X \\ &= UD^nX + D^{n-1}X + D^nX + (n-1)D^{n-1}X \\ &= UD^nX + D^nX + nD^{n-1}X. \end{aligned}$$

This completes the proof.

(d). Observing (c), it can be concluded that $s_{n+1} = s_n + ns_{n-1}$. We have

$$s_n = s_{n-1} + (n-1)s_{n-2}, n \geqslant 2,$$

where $s_0 = 1, s_1 = 1$.

(e). By (d), we have

$$F(x) = 1 + x + \sum_{n\geq 2} s_n \frac{x^n}{n!} = 1 + x + \sum_{n\geq 2} (s_{n-1} + (n-1)s_{n-2}) \frac{x^n}{n!}.$$

Now we take the derivative of $F(x)$ and obtain

$$F'(x) = 1 + \sum_{n\geq 2} s_{n-1} \frac{x^{n-1}}{(n-1)!} + x \sum_{n\geq 2} s_{n-2} \frac{x^{n-2}}{(n-1)!}$$

$$= (1+x)F(x).$$

We have

$$\frac{dF(x)}{dx} = (1+x)F(x),$$

i.e.,

$$\frac{dF(x)}{F(x)} = (1+x)dx.$$

By integration, we obtain:

$$\ln F(x) = x + \frac{x^2}{2} + c$$

Based on the initial conditions $F(0) = s_0 = 1$*, we have* $c = 0$*. So*

$$F(x) = e^{x+\frac{x^2}{2}}.$$

(f)*. Firstly, we calculate* s_n*. By*

$$F(x) = e^{x+\frac{x^2}{2}} = \sum_{m\geq 0} \frac{(x+\frac{x^2}{2})^m}{m!},$$

and the coefficient of x^n *is* $\frac{s_n}{n!}$*, we consider the coefficient of* x^n*. Let* $m = n-k$*,* $k = 0, 1, ..., \lfloor \frac{n}{2} \rfloor$*. Then the coefficient of* x^n *is determined by* $n-2k$*'s* x *and* k*'s* $\frac{x^2}{2}$*. We have*

$$\begin{aligned}
\frac{s_n}{n!} &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{1}{2^k (n-k)!} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-k)!}{k!(n-2k)!} \frac{1}{2^k (n-k)!} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^k k!(n-2k)!}.
\end{aligned}$$

So

$$s_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^k k!(n-2k)!}.$$

Now we consider the number of involutions in S_n*, i.e., the number of elements* $\pi \in S_n$ *satisfying* $\pi^2 = id$*. Suppose there are* k *2-cycles,* $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$*. There are* $\binom{n}{2k}$ *methods to select* $2k$ *elements. Put these* $2k$ *elements into* k *disjoint 2-cycles, we have*

$$1 \cdot 3 \cdot ... \cdot (2k-1) = \frac{(2k)!}{k!2^k}.$$

We have

$$\#\{\sigma, \sigma^2 = id\} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{2^k (k)!}$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(2k)!(n-2k)!} \frac{(2k)!}{2^k (k)!}$$
$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^k k!(n-2k)!}.$$

This completes the proof.

(g).........

(h). By $\pi^2 = id$, we have $\pi = \pi^{-1}$. By (g), $P = Q$, we have

$$s_n = \sum_{\lambda \vdash n} f^\lambda = \#\{\pi, \pi^2 = id\}.$$

Exercise 8.20. .

Exercise 8.21. .

Exercise 8.22. .

Exercise 8.23. .

Exercise 8.24. .

Exercise 8.25. (a). *The process constructed from the poset Z indicates that $\#Z_0 = 1, \#Z_1 = 1, \#Z_2 = 2$ and satisfies recursion relations*

$$\#Z_n = \#Z_{n-1} + \#Z_{n-2}, n \geqslant 2.$$

This recursive relationship is the recursive relationship of Fibonacci numbers, the initial condition is $\#Z_0 = 1, \#Z_1 = 1$. So we have $\#Z_n = F_{n+1}$. By the generating function of Fibonacci number, the rank-generating function of Z is given by

$$F(Z, q) = \frac{1}{1 - q - q^2}.$$

Note: By $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, ...,$ we have the generating function

$$\begin{aligned} F(x) &= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + ... \\ &= f_1 x + \sum_{i \geqslant 2} f_i x^i \\ &= x + \sum_{i \geqslant 2} (f_{i-2} + f_{i-1}) x^i \\ &= x + x^2 + \sum_{i \geqslant 2} f_{i-2} x^{i-2} + x \sum_{i \geqslant 2} f_{i-1} x^{i-1} \\ &= x + x^2 F(x) + x F(x). \end{aligned}$$

We have $F(x) = \frac{x}{1-x-x^2}$.

Similarly, by $f_0 = 1, f_1 = 1, f_2 = 2, f_3 = 3, ...,$ we have the generating function

$$\begin{aligned}F(x) &= f_0 + f_1x + f_2x^2 + f_3x^3 + ...\\ &= 1 + f_1x + \sum_{i\geqslant 2} f_ix^i\\ &= 1 + x + \sum_{i\geqslant 2}(f_{i-2} + f_{i-1})x^i\\ &= 1 + x + x^2F(x) + x(F(x) - 1)\\ &= 1 + x^2F(x) + xF(x).\end{aligned}$$

We have $F(x) = \frac{1}{1-x-x^2}$.

(b). *For $x \in Z_i$, we first take an upward walk on x and then a downward walk to obtain y. If $y \neq x$, we can obtain y for a downward walk of x and then a upward walk. Now, after the action of $D_{i+1}U_i - U_{i-1}D_i$ on x, the coefficient of y is 0. If $y = x$, it can be inferred from the "reflection" construction of Z. If the number of elements covered by x is r, then the coefficient of y after $U_{i-1}D_i$ acts on x is r.*

The number of elements covering x, in addition to the r elements reflected on it, there is also a new element added to cover x. The straight line drawn in Figure 8.1 (in book). So there are $r + 1$ elements covering x. The coefficient of y after $D_{i+1}U_i - U_{i-1}D_i$ acts on x is $(r + 1) - r = 1$. Therefore we have $D_{i+1}U_i - U_{i-1}D_i = I_i$.

Exercise 8.26. .

Exercise 8.27. .

Exercise 8.28. (a).........

(b).........

(c). *For $m, n \geqslant 1$, there are $mn + 1$ different integers in permutations $w \in G_{mn+1}$, denote $a_1a_2...a_{mn+1}$. Let m_i be the length that is the longest increasing subsequence starting from a_i. Let r_i be the length that is the longest decreasing subsequence starting from a_i. If $m_i \leqslant m, r_i \leqslant n$, then we construct the following mapping*

$$\varphi : a_i \longrightarrow (m_i, r_i).$$

This mapping is from $\{a_1, a_2, ..., a_{mn+1}\}$ to $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$. This is an injection. If $a_i \neq a_j$, then we have $r_i > r_j$ when $a_i > a_j$ and we have $r_i < r_j$ when $a_i < a_j$. We have $(m_i, r_i) \neq (m_j, r_j)$. However, we have

$$\#\{a_1, a_2, ..., a_{mn+1}\} = mn + 1 > \#\{1, 2, ..., m\} \times \{1, 2, ..., n\} = mn.$$

This is a contradiction.

(d)..........

Exercise 8.29. *The* 13 *plane partitions of* 4 *as follow.*

$$P_1 = 4, \quad P_2 = 31, \quad P_3 = 22, \quad P_4 = 211, \quad P_5 = 1111,$$

$$P_6 = \begin{array}{l} 3 \\ 1 \end{array}, \quad P_7 = \begin{array}{l} 2 \\ 2 \end{array}, \quad P_8 = \begin{array}{ll} 2 & 1 \\ 1 & \end{array}, \quad P_9 = \begin{array}{ll} 1 & 1 \\ 1 & 1 \end{array}, \quad P_{10} = \begin{array}{lll} 1 & 1 & 1 \\ 1 & & \end{array},$$

$$P_{11} = \begin{array}{l} 2 \\ 1 \\ 1 \end{array}, \quad P_{12} = \begin{array}{ll} 1 & 1 \\ 1 & \\ 1 & \end{array}, \quad P_{13} = \begin{array}{l} 1 \\ 1 \\ 1 \\ 1 \end{array}.$$

The 24 *plane partitions of* 5 *as follow.*

$$P_1 = 5, \quad P_2 = 41, \quad P_3 = 32, \quad P_4 = 311, \quad P_5 = 221, \quad P_6 = 2111, \quad P_7 = 11111,$$

$$P_8 = \begin{array}{l} 4 \\ 1 \end{array}, \quad P_9 = \begin{array}{l} 3 \\ 2 \end{array}, \quad P_{10} = \begin{array}{ll} 3 & 1 \\ 1 & \end{array}, \quad P_{12} = \begin{array}{ll} 2 & 1 \\ 1 & \end{array}, \quad P_{13} = \begin{array}{lll} 2 & 1 & 1 \\ 1 & & \end{array},$$

$$P_{14} = \begin{array}{ll} 2 & 1 \\ 1 & 1 \end{array}, \quad P_{15} = \begin{array}{llll} 1 & 1 & 1 & 1 \\ 1 & & & \end{array}, \quad P_{16} = \begin{array}{lll} 1 & 1 & 1 \\ 1 & & \end{array}, \quad P_{17} = \begin{array}{l} 3 \\ 1 \\ 1 \end{array}, \quad P_{18} = \begin{array}{l} 2 \\ 2 \\ 1 \end{array},$$

$$P_{19} = \begin{array}{ll} 2 & 1 \\ 1 & \\ 1 & \end{array}, \quad P_{20} = \begin{array}{lll} 1 & 1 & 1 \\ 1 & & \\ 1 & & \end{array}, \quad P_{21} = \begin{array}{ll} 1 & 1 \\ 1 & 1 \\ 1 & \end{array}, \quad P_{22} = \begin{array}{l} 2 \\ 1 \\ 1 \\ 1 \end{array}, \quad P_{23} = \begin{array}{ll} 1 & 1 \\ 1 & \\ 1 & \\ 1 & \end{array},$$

$$P_{24} = \begin{array}{l} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}.$$

Exercise 8.30. .

Exercise 8.31. .

Exercise 8.32. (a). *By*

$$\pi'(A) = \begin{pmatrix} 6 & 4 & 4 & 3 & 3 \\ 5 & 3 & 3 & 2 & \\ 3 & 2 & 1 & & \end{pmatrix},$$

we have

$$\pi'(A) = \begin{pmatrix} 5 & 5 & 5 & 3 & 1 & 1 \\ 4 & 4 & 3 & 1 & 1 & \\ 3 & 2 & 1 & & & \end{pmatrix}.$$

We have

$$P = \begin{matrix} 5\ 5\ 5\ 3\ 1\ 1 \\ 3\ 3\ 2 \\ 1 \end{matrix} \qquad Q = \begin{matrix} 3\ 3\ 3\ 2\ 2\ 1 \\ 2\ 1\ 1 \\ 1 \end{matrix},$$

$$\omega_A = \begin{pmatrix} 3 & 3 & 3 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\ 5 & 3 & 1 & 5 & 3 & 2 & 1 & 5 & 3 & 1 \end{pmatrix},$$

and

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}_{3\times 5}.$$

(b). The non-zero elements in A can construct an array ω_A. By the RSK algorithm, we can obtain the CSPP for (P, Q). We have

$$A \overset{RSK'}{\longleftrightarrow} (P, Q) \overset{\pi}{\longleftrightarrow} \pi(P, Q) \longleftrightarrow \pi'(P, Q).$$

Exercise 8.33. *.*

Exercise 8.34. *(a)........*

(b) The result can be referenced in the following book: David M. Bressoud, "Proofs and Confirmations: The story of the alternating sign matrix conjecture", Cambridge University Press, 1999. Please refer to Lemma 4.4 on page 165 and Equation 4.17 on page 138, which, when combined, yield the final result.

(c)..........

(d)...........

Exercise 8.35. *.*

Exercise 8.36. *Please refer to the book: David M. Bressoud, "Proofs and Confirmations: The story of the alternating sign matrix conjecture", Cambridge University Press, 1999. On page 13, Theorem 1.3 represents this generating function as:*

$$\sum_{n\geq 0} pp_{rst}(n)x^n = \prod_{i=1}^{r}\prod_{j=1}^{s} \frac{1-x^{i+j+t-1}}{1-x^{i+j-1}}.$$

In the 1970s, Ian Macdonald recognized that the generating function provided in Theorem 1.3 can also be formulated as

$$\sum_{n\geq 0} pp_{rst}(n)x^n = \prod_{i=1}^{r}\prod_{j=1}^{s} \frac{1-x^{i+j+t-1}}{1-x^{i+j-1}} = \prod_{i=1}^{r}\prod_{j=1}^{s}\prod_{k=1}^{t} \frac{1-x^{i+j+k-1}}{1-x^{i+j+k-2}}.$$

See page 14 of David M. Bressoud's book for more details.

Exercise 8.37. *.*

9 The Matrix-Tree Theorem

Exercise 9.1. *The number of spanning tree of the complete graph K_p is $\kappa(K_p) = p^{p-2}$. We consider the number of spanning tree of G_p.*

Method 1. Let's first label the vertices of K_p as $1, 2, ..., p$, and then remove one edge. There are $\frac{p(p-1)}{2}$ methods to remove one edge. Let S be the number of spanning tree of all G_p. Then we have

$$S = \kappa(K_p) \cdot \left(\frac{p(p-1)}{2} - (p-1) \right) = p^{p-2} \cdot \frac{(p-2)(p-1)}{2}.$$

The above equation can be understood as: A spanning tree in G_p is also a spanning tree in K_p, and the deleted edge does not affect this spanning tree. For example in K_4 as Figure 44. For

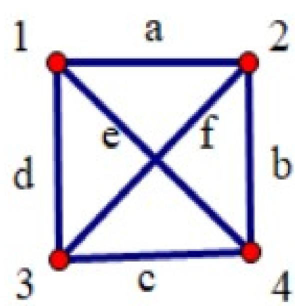

Figure 44:

the spanning tree deb, deleting any edge in a, f, c does not affect def.
Now we have the total sum of G_p's spanning trees. Because K_P is a complete graph of order p, where each edge is equivalent. There are a total of $\frac{p(p-1)}{2}$ edges, hence the number of spanning tree of G_T is:

$$\frac{S}{\frac{p(p-1)}{2}} = p^{p-2} \cdot \frac{(p-2)(p-1)}{2} \cdot \frac{2}{p(p-1)} = p^{p-3}(p-2).$$

The meaning of the above equation is: after removing one edge (all edges are equivalent) from the unlabelled K_p, and then label the vertices as $1, 2, ..., p$ to obtain the number of spanning trees for G_p.

Method 2. We use the Matrix-Tree Theorem. For G_p, suppose the removed edge as $1-2$. We have

$$L(G_p) = \begin{pmatrix} p-2 & 0 & -1 & -1 & ... & -1 & -1 \\ 0 & p-2 & -1 & -1 & ... & -1 & -1 \\ -1 & -1 & p-1 & -1 & ... & -1 & -1 \\ . & . & . & . & ... & . & . \\ . & . & . & . & ... & . & . \\ . & . & . & . & ... & . & . \\ -1 & -1 & -1 & -1 & ... & -1 & -1 \end{pmatrix}_{p \times p}.$$

After removing the first row and first column, we obtain $L_0(G_p)$:

$$|L_0(G_p)| = \begin{bmatrix} p-2 & 0 & 0 & ... & 0 & -p \\ 0 & p & 0 & ... & 0 & -p \\ 0 & 0 & p & ... & 0 & -p \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & p & -p \\ -1 & -1 & -1 & ... & -1 & p-1 \end{bmatrix}_{(p-1)\times(p-1)}$$

$$= \begin{bmatrix} p-1 & 0 & 0 & ... & 0 & -1 \\ 0 & p & 0 & ... & 0 & 0 \\ 0 & 0 & p & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & p & 0 \\ -1 & -1 & -1 & ... & -1 & p1 \end{bmatrix}_{(p-1)\times(p-1)}$$

$$= (p-1)p^{p-3} + (-1)^p(-1)\cdot(-1)^{p-1}(-1)p^{p-3}$$

$$= (p-2)p^{p-3}.$$

Therefore we have $\kappa(G_p) = (p-2)p^{p-3}$.

Exercise 9.2. *The $K_{r,s}$ is the complete bipartite graph. We have*

$$L = \begin{pmatrix} s & 0 & 0 & ... & 0 & -1 & -1 & ... & -1 \\ 0 & s & 0 & ... & 0 & -1 & -1 & ... & -1 \\ . & . & . & ... & . & . & . & ... & . \\ . & . & . & ... & . & . & . & ... & . \\ 0 & 0 & 0 & ... & s & -1 & -1 & ... & -1 \\ -1 & -1 & -1 & ... & -1 & r & 0 & ... & 0 \\ -1 & -1 & -1 & ... & -1 & 0 & r & ... & 0 \\ . & . & . & ... & . & . & . & ... & . \\ . & . & . & ... & . & . & . & ... & . \\ -1 & -1 & -1 & ... & -1 & 0 & 0 & ... & r \end{pmatrix}_{(r+s)\times(r+s)},$$

Then we have

$$L - rI = \begin{pmatrix} s-r & 0 & 0 & ... & 0 & -1 & -1 & ... & -1 \\ 0 & s-r & 0 & ... & 0 & -1 & -1 & ... & -1 \\ . & . & . & ... & . & . & . & ... & . \\ . & . & . & ... & . & . & . & ... & . \\ 0 & 0 & 0 & ... & s-r & -1 & -1 & ... & -1 \\ -1 & -1 & -1 & ... & -1 & 0 & 0 & ... & 0 \\ -1 & -1 & -1 & ... & -1 & 0 & 0 & ... & 0 \\ . & . & . & ... & . & . & . & ... & . \\ . & . & . & ... & . & . & . & ... & . \\ -1 & -1 & -1 & ... & -1 & 0 & 0 & ... & 0 \end{pmatrix}_{(r+s)\times(r+s)}.$$

By performing linear transformation on $L-rI$, we can obtain:

$$L-rI \to \begin{pmatrix} s-r & 0 & 0 & ... & 0 & -1 & -1 & ... & -1 \\ 0 & s-r & 0 & ... & 0 & -1 & -1 & ... & -1 \\ . & . & . & ... & . & . & . & ... & . \\ . & . & . & ... & . & . & . & ... & . \\ 0 & 0 & 0 & ... & s-r & -1 & -1 & ... & -1 \\ -1 & -1 & -1 & ... & -1 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 & ... & 0 \\ . & . & . & ... & . & . & . & ... & . \\ . & . & . & ... & . & . & . & ... & . \\ 0 & 0 & 0 & ... & 0 & 0 & 0 & ... & 0 \end{pmatrix}_{(r+s)\times(r+s)} .$$

Taking the determinant of $L-xI$, we can obtain:

$$\begin{aligned} |L-xI| &= \begin{bmatrix} x-s & 0 & 0 & ... & 0 & 1 & 1 & ... & 1 \\ 0 & x-s & 0 & ... & 0 & 1 & 1 & ... & 1 \\ . & . & . & ... & . & . & . & ... & . \\ . & . & . & ... & . & . & . & ... & . \\ 0 & 0 & 0 & ... & x-s & 1 & 1 & ... & 1 \\ 1 & 1 & 1 & ... & 1 & x-r & 0 & ... & 0 \\ 1 & 1 & 1 & ... & 1 & 0 & x-r & ... & 0 \\ . & . & . & ... & . & . & . & ... & . \\ . & . & . & ... & . & . & . & ... & . \\ 1 & 1 & 1 & ... & 1 & 0 & 0 & ... & x-r \end{bmatrix}_{(r+s)\times(r+s)} \\ &= \begin{bmatrix} x-s & 0 & 0 & ... & 0 & 1 & 1 & ... & 1 \\ 0 & x-s & 0 & ... & 0 & 1 & 1 & ... & 1 \\ . & . & . & ... & . & . & . & ... & . \\ . & . & . & ... & . & . & . & ... & . \\ 0 & 0 & 0 & ... & x-s & 1 & 1 & ... & 1 \\ 0 & 0 & 0 & ... & 0 & x-r & 0 & ... & r-x \\ 0 & 0 & 0 & ... & 0 & 0 & x-r & ... & r-x \\ . & . & . & ... & . & . & . & ... & . \\ . & . & . & ... & . & . & . & ... & r-x \\ 1 & 1 & 1 & ... & 1 & 0 & 0 & ... & x-r \end{bmatrix}_{(r+s)\times(r+s)} \\ &= \begin{bmatrix} x-s & 0 & 0 & ... & 0 & 1 & 1 & ... & s \\ 0 & x-s & 0 & ... & 0 & 1 & 1 & ... & s \\ . & . & . & ... & . & . & . & ... & . \\ . & . & . & ... & . & . & . & ... & . \\ 0 & 0 & 0 & ... & x-s & 1 & 1 & ... & s \\ 0 & 0 & 0 & ... & 0 & x-r & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & x-r & ... & 0 \\ . & . & . & ... & . & . & . & ... & . \\ . & . & . & ... & . & . & . & ... & . \\ 1 & 1 & 1 & ... & 1 & 0 & 0 & ... & x-r \end{bmatrix}_{(r+s)\times(r+s)} , \end{aligned}$$

and

$$|L - xI| = (x-r)^{s-1} \begin{bmatrix} x-s & 0 & 0 & \ldots & 0 & s \\ 0 & x-s & 0 & \ldots & 0 & s \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \ldots & x-s & s \\ 1 & 1 & 1 & \ldots & 1 & x-r \end{bmatrix}$$

$$= (x-r)^{s-1} \begin{bmatrix} x-s & 0 & 0 & \ldots & 0 & x \\ 0 & x-s & 0 & \ldots & 0 & x \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \ldots & x-s & x \\ 1 & 1 & 1 & \ldots & 1 & x \end{bmatrix}$$

$$= x(x-r)^{s-1} \begin{bmatrix} x-s & 0 & 0 & \ldots & 0 & 1 \\ 0 & x-s & 0 & \ldots & 0 & 1 \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \ldots & x-s & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 \end{bmatrix}$$

$$= x(x-r)^{s-1} \begin{bmatrix} x-s & 0 & 0 & \ldots & 0 & 0 \\ 0 & x-s & 0 & \ldots & 0 & 0 \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \ldots & x-s & 0 \\ 1 & 1 & 1 & \ldots & 1 & (1-\frac{r}{x-s}) \end{bmatrix}$$

$$= x(x-r)^{s-1}(x-s)^r(1-\frac{r}{x-s})$$

$$= x(x-r)^{s-1}(x-s)^{r-1}(x-s-r).$$

(a). We have $rank(L - rI) \leqslant r+1$*. A lower bound on the number of eigenvalues of* L *equal to* r *is* $\geqslant s-1$*.*

(b). Assume $s \neq r$*. By the symmetry of* s *and* r*, the number of eigenvalues of* L *equal to* s *is* $r-1$*.*

(c). By $|L - xI| = x(x-r)^{s-1}(x-s)^{r-1}(x-s-r)$*, the eigenvalues of* L *are*

$$0 \ (number:\ 1),\ r\ (number:\ s-1),\ s\ (number:\ r-1),\ r+s,\ (number:\ 1).$$

Note: The author does not seem to want to directly obtain all the eigenvalues of L *from the equation of* $|L - xI|$*. By (a) and (b), the number of eigenvalues of* L *equal to* r *is* $\geqslant s-1$*. The number of eigenvalues of* L *equal to* s *is* $\geqslant r-1$*. We have* $tr(L) = 2rs$*. The sum of all rows in* L *is* 0*. So we have* $|L| = 0$ *when* $x = 0$ *in* $|L - xI|$*. There is a eigenvalue 0 of* L*. There exist* $r+s$ *eigenvalues of* L*. (now at least* $(s-1)'$ r*,* $(r-1)'$ s *and* $1'$ *0.) Then the other eigenvalue is* $2rs - 0 - r(s-1) - s(r-1) = r+s$*.*

(d). *We know all the eigenvalues of L. We have*

$$\kappa(k_{rs}) = \frac{1}{r+s} r^{s-1} s^{r-1} (r+s) = r^{s-1} s^{r-1}.$$

(e) Let $V(K_{rs}) = V_1 \cup V_2$, where $V_1 = \{u_1, ..., u_r\}$ and $V_2 = \{v_1, ..., v_s\}$. Let T be a spanning tree of K_{rs}. For any $e \in E(T)$ satisfies $e = \{u_k, v_r\}$, we delate the largest leaf. If the largest leaf belongs to V_1, then record its parent node in C_2. If the largest leaf belongs to V_2, then record its parent node in C_1. Until there are two remaining vertices, one of them must belong to V_1, another belonging to V_2. So $C_1 \in V_1^{s-1}$, $C_2 \in V_2^{r-1}$. Therefore we have

$$\#\{C_1\} \cdot \#\{C_2\} = r^{s-1} s^{r-1}.$$

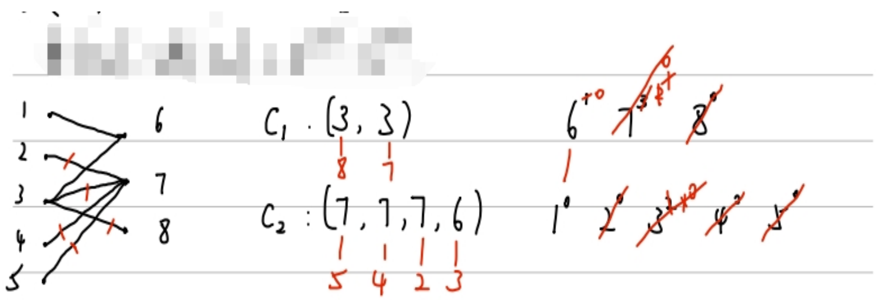

Figure 45:

Exercise 9.3. .

Exercise 9.4. *By the definition of G_p, we have Figure 46.*

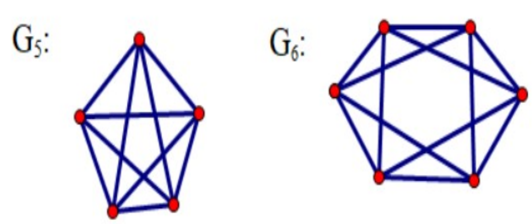

Figure 46:

The eigenvalues see Exercise 5(h) for Chapter 2.

The G_P is 4 degree regular. Consider the eigenvalues of $A(Z_p)$

$$\xi_j^2 + \xi_j^{-2} + \xi_j + \xi_j^{-1},$$

where $\xi_j = e^{\frac{2\pi i j}{p}}$. *Firstly, we have*

$$\prod_{j=0}^{p-1}(x-\xi_j) = x^p - 1, \quad \prod_{j=1}^{p-1}(x-\xi_j) = \frac{x^p-1}{x-1}.$$

If $x = 0$, *then we have*

$$\prod_{j=1}^{p-1}(-\xi_j) = (-1)^{p-1}\prod_{j=1}^{p-1}\xi_j = 1 \implies \prod_{j=1}^{p-1}\xi_j = (-1)^{p-1}.$$

If $x = 1$, *then we have*

$$\prod_{j=1}^{p-1}(1-\xi_j) = p.$$

We have

$$\begin{aligned}
\kappa(G) &= \frac{1}{p}\prod_{j=1}^{p-1}(4-(\xi_j^2+\xi_j^{-2}+\xi_j+\xi_j^{-1})) \\
&= \frac{1}{p}\prod_{j=1}^{p-1}(-\xi_j^{-2})(\xi_j^2+3\xi_j+1)(\xi_j-1)^2 \\
&= p\cdot(-1)^p\prod_{j=1}^{p-1}(\xi_j^2+3\xi_j+1) \\
&= p\cdot(-1)^p\prod_{j=1}^{p-1}(\xi_j+a^2)(\xi_j+b^2) \\
&= p\cdot(-1)^p\frac{(-a^2)^p-1}{-a^2-1}\cdot\frac{(-b^2)^p-1}{-b^2-1} \\
&= p\cdot(-1)^{p-1}\cdot\frac{1}{5}((-a^2)^p-1)((-b^2)^p-1) \\
&= p\cdot\frac{1}{5}(-1)^{p-1}((-1)^{p+1}a^{2p}+(-1)^{p+1}b^{2p}+2) \\
&= p\cdot\frac{1}{5}(a^{2p}+b^{2p}-2(-1)^p) \\
&= p\cdot\frac{1}{5}(a^p-b^p)^2 \\
&= p\cdot F_p^2,
\end{aligned}$$

where $a^2b^2 = 1$, $-(a^2+b^2) = -3$, $ab = -1$, $a+b = 1$,

$$F_p = \frac{1}{\sqrt{5}}(a^p - b^p).$$

Exercise 9.5. *The* C_n *is the* n*-cube. It is* n *degree regular. The* C_n *have* $\binom{n}{i}$ *eigenvalues equal to* $n-2i$, *for each* $0 \leqslant i \leqslant n$. *Let* K_{2^n} *be the complete graph of order* 2^n. *It is* $2^n - 1$ *degree* *regular. The eigenvalues are* -1 *(number :* $(2^n - 1)$*),* $2^n - 1$ *(number :* 1*). By* $\overline{C_n} = K_{2^n} - C_n$ *and the symmetry of* K_{2^n}, C_n*, the* $\overline{C_n}$ *is* $2^n - n - 1$ *degree regular. The adjacency matrix of* $\overline{Cn}$ *is the adjacency matrix of* K_{2^n} *minus the adjacency matrix of* C_n*. Therefore the eigenvalues of* $\overline{C_n}$ *are*

$$-1 - (n - 2i) \quad \left(number: \quad \binom{n}{i} \right), \quad 1 \leqslant i \leqslant n; \quad 2^n - n - 1 \ (number: \ 1).$$

We have

$$\begin{aligned} \kappa(\overline{C_n}) &= \frac{1}{2^n} \prod_{i=1}^{n} (2^n - 2i)^{\binom{n}{i}} \\ &= 2^{2^n - n - 1} \prod_{i=1}^{n} (2^{n-1} - i)^{\binom{n}{i}}. \end{aligned}$$

Note: $\prod_{i=1}^{n} 2^{\binom{n}{i}} = 2^{2^n - 1}$.

Exercise 9.6. .

Exercise 9.7. (*a*). *We have*

$$A = \begin{pmatrix} 0 & M \\ M^t & 0 \end{pmatrix}, \ L = \begin{pmatrix} a & & & & & & & & \\ & a & & & & & -M & & \\ & & \dots & & & & & & \\ & & & a & & & & & \\ & & & & b & & & & \\ & & & & & b & & & \\ & -M^t & & & & & \dots & & \\ & & & & & & & b \end{pmatrix}.$$

From the multiplication of block matrices, it can be concluded that:

$$
\begin{aligned}
A^2 &= \begin{pmatrix} 0 & M \\ M^t & 0 \end{pmatrix} \begin{pmatrix} 0 & M \\ M^t & 0 \end{pmatrix} = \begin{pmatrix} MM^t & 0 \\ 0 & M^tM \end{pmatrix} \\
&= \begin{pmatrix} 0 & -M \\ -M^t & (b-a)I \end{pmatrix} \begin{pmatrix} (a-b)I & -M \\ -M^t & 0 \end{pmatrix} \\
&= \begin{pmatrix} a-a & & & & & & & \\ & a-a & & & & & -M & \\ & & ... & & & & & \\ & & & a-a & & & & \\ & & & & b-a & & & \\ & & & & & b-a & & \\ & -M^t & & & & & ... & \\ & & & & & & & b-a \end{pmatrix} \\
&\quad . \begin{pmatrix} a-b & & & & & & & \\ & a-b & & & & & -M & \\ & & ... & & & & & \\ & & & a-b & & & & \\ & & & & b-b & & & \\ & & & & & b-b & & \\ & -M^t & & & & & ... & \\ & & & & & & & b-b \end{pmatrix} \\
&= (L-aI)(L-bI).
\end{aligned}
$$

The eigenvalues of $L(G)$ are $\lambda_1, \lambda_2, ..., \lambda_p$. Therefore, there exists an invertible matrix H, such that:

$$
HLH^{-1} = \begin{pmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 0 & \lambda_2 & 0 & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & \lambda_p \end{pmatrix}.
$$

We have

$$
H(L-aI)H^{-1} = \begin{pmatrix} \lambda_1 - a & 0 & 0 & ... & 0 \\ 0 & \lambda_2 - a & 0 & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & \lambda_p - a \end{pmatrix}
$$

$$
H(L-bI)H^{-1} = \begin{pmatrix} \lambda_1 - b & 0 & 0 & ... & 0 \\ 0 & \lambda_2 - b & 0 & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & \lambda_p - b \end{pmatrix}.
$$

We have

$$\begin{aligned} HA^2H^{-1} &= H(L-aI)(L-bI)H^{-1} \\ &= \begin{pmatrix} (\lambda_1-a)(\lambda_1-b) & 0 & 0 & ... & 0 \\ 0 & (\lambda_2-a)(\lambda_2-b) & 0 & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & (\lambda_p-a)(\lambda_p-b) \end{pmatrix}. \end{aligned}$$

The eigenvalues of A^2 *are* $(\lambda_1-a)(\lambda_1-b),\ (\lambda_2-a)(\lambda_2-b),\ ...,\ (\lambda_p-a)(\lambda_p-b)$.

(b). *See Figure 47.*

Exercise 9.8. (a). *Lemma: Two commutative* $p \times p$ *matrices* A*and* B *can be diagonalized simultaneously, i.e., there exists an invertible matrix* X *such that* XAX^{-1} *and* XBX^{-1} *are both upper triangular matrices.*
Proof: We use mathematical induction method. For $n = 1$*, the conclusion is clearly valid. Assuming the lemma holds for* $n < k$*. For* $n = k$*, let the eigenvalue of* A*be* λ*, and the corresponding characteristic subspace be* W*. By* $AB = BA$*, for any* $x \in W$*, we have* $A(Bx) = B(Ax) = \lambda Bx$*. So* $Bx \in W$ *and* W *is an invariant subspace of* B*. We take a set of bases of* W *and expand it into a set of bases of the entire space. The matrix with this set of bases as column vectors is* S*. We have*

$$C = S^{-1}AS = \begin{pmatrix} \lambda E & * \\ 0 & G \end{pmatrix}, \quad D = S^{-1}BS = \begin{pmatrix} F & * \\ 0 & H \end{pmatrix}.$$

From A *and* B *being commutative, we know that* S *and* D *can be commutative, and then* G *and* H *can be commutative. By the order of* G *and* H *are* $< k$*. There exists an invertible matrix* T *such that* $K = T^{-1}GT, L = T^{-1}HT$ *is an upper triangular matrix. There exists an invertible matrix* R *such that* $J = R^{-1}FR$ *is an upper triangular matrix. Let*

$$Q = \begin{pmatrix} R & 0 \\ 0 & T \end{pmatrix}.$$

By Q *is an invertible matrix, we know that*

$$Q^{-1}CQ = \begin{pmatrix} \lambda E & * \\ 0 & K \end{pmatrix}, \quad Q^{-1}DQ = \begin{pmatrix} J & * \\ 0 & L \end{pmatrix}$$

are upper triangular matrices. Let $X = SQ$*,* $X^{-1}AX$ *and* $X^{-1}BX$ *are both upper triangular matrices. This completes the proof.*

Now we consider (a)*. For the matrix of all* 1*'s, the eigenvalues of the matrix are as follows: an eigenvalue of* 0 *with multiplicity* $p-1$ *and an eigenvalue of* p *with multiplicity one. The* $L(G)$ *is a real symmetric matrix with the column sum 0. We have* $L \cdot J = J \cdot L = 0$*. Therefore, there exists an invertible matrix* X *that allows* L*and* αJ *to be simultaneously up triangulated. We add all rows of* L *to the last row and make the first* $p-1$*column of the last row* 0*. We have*

$$L \to \begin{pmatrix} & & * & & \\ & & & & \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}.$$

5.(a) $A(G)=\begin{pmatrix}0 & A_1\\ A_1^t & 0\end{pmatrix}$, $L(G)=\begin{pmatrix}aI_a & -A_1\\ -A_1^t & bI_b\end{pmatrix}$, 则 $A^2=(L-a\mathbb{1})(L-b\mathbb{1})$.

(b) A: $k-1$个1, $\#A=\binom{n}{k-1}$, $\deg=n-k+1=a$ $\quad (n-k+1)\binom{n}{k-1}=k\binom{n}{k}$

B: k个1, $\#B=\binom{n}{k}$, $\deg=k=b$

$A^2(G_{n,k})$ 的特征值为 0 ($\binom{n}{k}-\binom{n}{k-1}$ 重). $i(n-2k+i+1)$ ($2\binom{n}{k-i}-2\binom{n}{k-i-1}$ 重) $1\le i\le k$

$A^2(G_{n,k})$ 有特征值 $k(n-k+1)$ (2重) $\lambda_p=0, \lambda_{p-1}=n+1$

$k\le\frac{n}{2}$, $k-1$个1 或 k个1 形成的子图.

$(\lambda-a)(\lambda-b)=i(n-2k+i+1)$, $1\le i<k$. 两根 λ_1,λ_2:

$\lambda_1\lambda_2=(k-i)(n-k+i+1)$, $\lambda_1+\lambda_2=a+b=n+1$.

设 $L(G_{n,k})$ 余下的特征值为 x个a 和 y个b. 则

$$\begin{cases} x+y=\binom{n}{k}-\binom{n}{k-1}\\ xa+yb=a^2+b^2-(a+b)-(a+b)\sum_{i=1}^{k-1}\left(\binom{n}{k-i}-\binom{n}{k-i-1}\right)=a^2+b^2-(a+b)\binom{n}{k-1}\end{cases}$$

$$\Rightarrow\begin{cases} x=\frac{1}{b-a}\left(a\binom{n}{k-1}+b\binom{n}{k}-a^2-b^2\right)\\ y=\frac{1}{a-b}\left(a\binom{n}{k}+b\binom{n}{k-1}-a^2-b^2\right)\end{cases}$$

$$K(G_{n,k})=\frac{n+1}{\binom{n}{k}+\binom{n}{k-1}}\cdot\prod_{i=1}^{k-1}\left((k-i)(n-k+i+1)\right)^{\binom{n}{k-i}-\binom{n}{k-i-1}}\cdot a^x\cdot b^y$$

Figure 47:

Similarly, for αJ, we have

$$\alpha J \to \begin{pmatrix} \alpha & \alpha & \alpha & ... & \alpha \\ . & . & . & ... & . \\ . & . & . & ... & . \\ \alpha & \alpha & \alpha & ... & \alpha \\ \alpha p & \alpha p & \alpha p & ... & \alpha p \end{pmatrix} \to \begin{pmatrix} & & * & & \\ & & & & \\ 0 & 0 & 0 & ... & \alpha p \end{pmatrix}.$$

There exists an invertible matrix X that

$$X^{-1}LX = \begin{pmatrix} \theta_1 & & & & & \\ 0 & \theta_2 & & & * & \\ . & . & . & & & \\ . & . & . & .. & & \\ 0 & 0 & 0 & ... & \theta_{p-1} & \\ 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix}, \quad X^{-1}(\alpha J)X = \begin{pmatrix} 0 & & & & & \\ 0 & 0 & & & * & \\ . & . & . & & & \\ . & . & . & .. & & \\ 0 & 0 & 0 & ... & 0 & \\ 0 & 0 & 0 & ... & 0 & \alpha p \end{pmatrix},$$

and

$$X^{-1}(L+\alpha J)X = \begin{pmatrix} \theta_1 & & & & & \\ 0 & \theta_2 & & & * & \\ . & . & . & & & \\ . & . & . & .. & & \\ 0 & 0 & 0 & ... & \theta_{p-1} & \\ 0 & 0 & 0 & ... & 0 & \alpha_p \end{pmatrix}.$$

Therefore the eigenvalues of $L+\alpha J$ are $\theta_1, \theta_2, ..., \theta_{p-1}, \alpha p$.

(b). By K_p is a complete graph of order p, the laplacian matrix of K_p is

$$L(K_p) = \begin{pmatrix} p-1 & -1 & -1 & ... & -1 \\ -1 & p-1 & -1 & ... & -1 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ -1 & -1 & -1 & ... & p-1 \end{pmatrix}$$
$$= PI - J = PI + \alpha J,$$

where $\alpha = -1$, I is the identity matrix, and J is the matrix of all 1's. Therefore we have

$$L(G \cup K_p) = L(G) + L(K_p) = L + \alpha J + pI.$$

By (a), the eigenvalues of $L + \alpha J$ are $\theta_1, \theta_2, ..., \theta_{p-1}, \alpha p$. The PI is a diagonal matrix and is commutative with $L + \alpha J$. The eigenvalues of PI are p with multiplicity p. Therefore the eigenvalues of $L(G \cup K_p)$ are $\theta_1 + p, \theta_2 + p, ..., \theta_{p-1} + p, 0$. We have

$$\kappa(G \cup K_p) = \frac{1}{p}(\theta_1 + p)(\theta_2 + p)...(\theta_{p-1} + p) = \frac{1}{p}\prod_{i=1}^{p-1}(\theta_i + p).$$

(*c*). *We have* $G \cup \overline{G} = K_p$*, where* K_p *is a complete graph of order* p*. So we have*

$$L(G) + L(\overline{G}) = L(K_p) = pI - J.$$

We have $L(\overline{G}) = pI - (L(G) + J)$*. The eigenvalues of* $L(G) + J$ *are* $\theta_1, \theta_2, ..., \theta_{p-1}, p$*. Similarly, by (a), the eigenvalues of* $L(\overline{G})$ *are* $p - \theta_1, p - \theta_2, ..., p - \theta_{p-1}, 0$*. We have*

$$\kappa(\overline{G}) = \frac{1}{p}(p - \theta_1)(p - \theta_2)...(p - \theta_{p-1}) = \frac{1}{p}\prod_{i=1}^{p-1}(p - \theta_i).$$

(*d*). *See Figure 48.*

(d) $P(G,x) = \sum_{j=1}^{p} f_j(G)\, x^{j-1}$，其中 $f_j(G)$ 表示G具有j个连通分支的生成有根森林的个数.

由8(a), $\det(L(G) - xI) = -x(\theta_1 - x)\cdots(\theta_{p-1} - x) = \sum_{j=1}^{p}(-1)^j f_j(G)\, x^j$

$\Rightarrow (\theta_1 - x)\cdots(\theta_{p-1} - x) = \sum_{j=1}^{p}(-1)^{j-1} f_j(G)\, x^{j-1} = P(G,-x)$

$\Rightarrow P(\overline{G},-x) = (p - \theta_1 - x)\cdots(p - \theta_{p-1} - x)$

$= (-1)^{p-1}(\theta_1 + x - p)\cdots(\theta_{p-1} + x - p)$

$= (-1)^{p-1} P(G, x - p)$

$\Rightarrow P(\overline{G}, x) = (-1)^{p-1} P(G, -x - p)$.

Figure 48:

Exercise 9.9. *The* G^* *is the dual graph of* G*. The* G' *is graph that the "outside" vertex deleted. For example, we can see Figure 49.*

Therefore we have
(1). $\#E(G) = \#E(G^*)$.
(2). *In* G^**, except for outside vertices, the degree of all other vertices is* 4.
(3). *The* G^* *has one more vertex than* G'*, i.e., the outside vertex.*
(4). $\kappa(G) = \kappa(G^*)$*, Corollary 11.19. (Proof required ???)*
The eigenvalues of adjacency matrix $A(G')$ *are* $\lambda_1, \lambda_2, ..., \lambda_p$*. There exists an invertible matrix* X *that*

$$T^{-1}A(G')T = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & ... & & \\ & & ... & & \\ & & & & \lambda_p \end{pmatrix}.$$

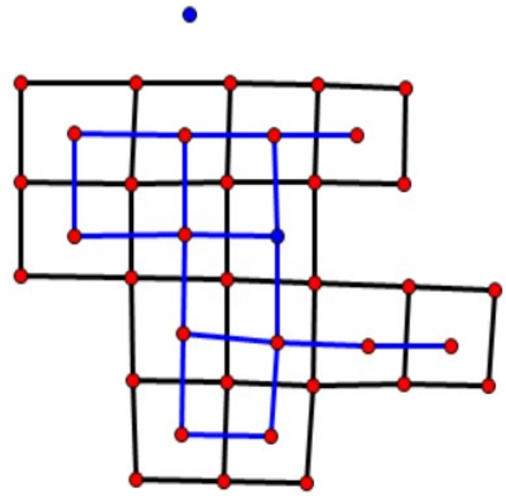

Figure 49:

We have

$$L(G^*) = \begin{pmatrix} 4 & & & & & * \\ & 4 & & M & & * \\ & & 4 & & & * \\ & M^T & & .. & & * \\ & & & & 4 & * \\ * & * & * & & ... & \triangle \end{pmatrix}_{(p+1)\times(p+1)} .$$

Let $L_0(G*)$ be the matrix obtained by removing the last row and last column. We have

$$L_0(G^*) = \begin{pmatrix} 4 & & & & \\ & 4 & & M & \\ & & 4 & & \\ & M^T & & .. & \\ & & & & 4 \end{pmatrix}_{p\times p} = 4T_{p\times p} - A(G^*).$$

Therefore we have

$$\begin{aligned} |L_0(G^*)| &= |T^{-1}L_0(G^*)T| = |4T^{-1}IT - T^{-1}A(G')T| \\ &= \begin{bmatrix} 4-\lambda_1 & & & \\ & 4-\lambda_2 & & \\ & & ... & \\ & & & 4-\lambda_p \end{bmatrix} \\ &= \prod_{i=1}^{P}(4-\lambda_i). \end{aligned}$$

By the Matrix-Tree Theorem, we have

$$\kappa(G) = \kappa(G^*) = |L_0(G*)| = \prod_{i=1}^{p}(4 - \lambda_i).$$

Exercise 9.10. *(a).........*

(b). Incomplete.......
We have $det(L - xI) = -x(\mu_1 - x)(\mu_2 - x)...(\mu_{p-1} - x)$. *The sign of the coefficient of* x^j *is* $(-1)^j$. *We have*

$$|L - xI| = \begin{bmatrix} d_1 - x & & & M \\ & d_2 - x & & \\ & & ... & \\ & M^t & & d_p - x \end{bmatrix} = \begin{bmatrix} \mu_1 - x & & & \\ & \mu_2 - x & & \\ & & ... & \\ & & & \mu_p - x \end{bmatrix}.$$

By $1 \leqslant j \leqslant p$, *we have*
If $j = 1$, *then* $f_1(G) = \mu_1\mu_2...\mu_p = p\kappa(G)$. *There is a connected component. The all* p *vertices can as roots.*
If $j = 2$, *then* $f_2(G) = \sum_{i=1}^{p-1} \mu_1\mu_2...\widehat{\mu_i}...\mu_p = p\kappa(G)$.
...
If $j = p - 1$, *then* $f_{p-1}(G) = \mu_1 + \mu_2 + ... + \mu_{p-1}$.
If $j = p$, *then* $f_p(G) = 1$. *There are* p *connected components. Each vertex is a root, and there is only one case.*

(c). We consider $p(x) = \sum_{i=1}^{p} f_i(G)x^i$. *This polynomial is obviously a real coefficient polynomial. We have*

$$\begin{aligned} p(-x) &= -f_1(G)x + f_2(G)x^2 - f_3(G)x^3 + ... + (-1)^p f_p(G)x^p \\ &= |L - xI| = det|L - xI|. \end{aligned}$$

The L *is the laplacian matrix of graph* G. *This is a real symmetric matrix. Therefore, their eigenvalues are all real numbers. The roots of* $p(-x) = det|L - xI| = 0$ *are all real roots.* $(\mu_1, \mu - 2, ..., \mu_{p-1}, 0.)$ *Therefore, the roots of* $p(x) = 0$ *are* $-\mu_1, -\mu - 2, ..., -\mu_{p-1}, 0$, *and their are also real roots. By Theorem 5.12, the sequence* $f_1(G)$, $f_2(G)$, ..., $f_p(G)$ *is strongly log-concave.*

(d).........

Exercise 9.11. .

Exercise 9.12. .

Exercise 9.13. .

Exercise 9.14. *If we have $f_S(p) = kp^{p-k-1}$, then the number of $f_k(p)$ of the planted forests on the vertex set $[p]$ with exactly k components can be obtained in two steps. Firstly, we select a k-element subset S from $[p]$. There are $\binom{p}{k}$ options available. Secondly, we count the number of planted forests with exactly k components, whose set of roots is S. We have*

$$
\begin{aligned}
f_k(k) &= \binom{p}{k} \cdot f_S(p) = k\binom{p}{k} p^{p-k-1} \\
&= \frac{p!k}{k!(p-k)!}\frac{1}{p} \cdot p^{p-k} \\
&= \binom{p-1}{k-1} p^{p-k}.
\end{aligned}
$$

Now we need to prove that $f_S(p) = k \cdot p^{p-k-1}$.

See Figure 50.

12 $S \subseteq [p]$, $\#S = k$.

$f_S(p)$：$[p]$上的有k个连通分支且根集为S的有根森林的个数.

$f_k(p)$：$[p]$上的恰有k个连通分支的有根森林的个数.

pf. $f_S(p) = kp^{p-k-1}$. 对给定的S，令$\tilde{P}$：$[p]$上的至多有k个连通分支且根集为S的子集的有根森林的集合.

约定：顶层为第1层. 则第j层的元素有$p-j$条边，$1 \le j \le k$.

两种方法计数极大链的条数.

(1) 底层.

$$\left.\begin{aligned} &f_S(p) \cdot p(k-1) \cdot p(k-2) \cdot \cdots \cdot p \cdot 1 = f_S(p) \cdot p^{k-1} \cdot (k-1)! \\ &\text{(2) 顶层.} \\ &k \cdot \kappa(K_p) \cdot (k-1) \cdot (k-2) \cdots 1 = k \cdot p^{p-2} \cdot (k-1)! \end{aligned}\right\} \Rightarrow f_S(p) = k \cdot p^{p-k-1}.$$

设$F \in \tilde{P}$有j个连通分支，根集为S'. $\#S' = j$

选择边uv使删掉uv并将v作为新分支的根的森林$F' \in \tilde{P}$. 即$v \in S \setminus S'$，则v有$k-j$种选择. 而u由v唯一确定. 故F覆盖$k-j$个元素.

$f_k(p) = \binom{p}{k} \cdot f_S(p) = k\binom{p}{k} p^{p-k-1} = p\binom{p-1}{k-1} \cdot p^{p-k-1} = \binom{p-1}{k-1} p^{p-k}$.

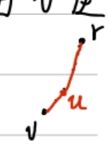

Figure 50:

10 Eulerian Digraphs and Oriented Trees

Exercise 10.1. (a). *Method 1: (A direct argument) Because D is connected and balanced, D is Eulerian digraphs. The number of $\tau(D,v)$ is independent of the selection of v. Let $v = v_1$. A directed tree with v_1 as its root can only be*

$$v_p \to v_{p-1} \to ... \to v_2 \to v_1.$$

The $v_2 \to v_1$ has a_1 options to choose from, ..., The $v_{i+1} \to v_i$ has a_i options to choose from, $1 \leqslant i \leqslant p-1$. Therefore we have $\tau(D,v) = \tau(D,v_1) = a_1a_2...a_{p-1}$.

Method 2: (Using determinants) The laplacian matrix of digraphs is

$$L(D) = \begin{pmatrix} a_1 & a_2 & & & & & & \\ -a_1 & a_1+a_2 & -a_2 & & & & & \\ & -a_2 & a_2+a_3 & -a_3 & & & & \\ & & & ... & & & & \\ & & & & ... & & & \\ & & & & & -a_{p-2} & a_{p-2}+a_{p-1} & -a_{p-1} \\ & & & & & & -a_{p-1} & a_{p-1} \end{pmatrix}_{p\times p}.$$

Therefore, we have

$$\begin{aligned} \tau(D,v) &= |L_0(D)| \\ &= \begin{bmatrix} a_1 & a_2 & & & & & \\ -a_1 & a_1+a_2 & -a_2 & & & & \\ & -a_2 & a_2+a_3 & -a_3 & & & \\ & & & ... & & & \\ & & & & ... & & \\ & & & & & -a_{p-2} & a_{p-2}+a_{p-1} \end{bmatrix} \\ &= \begin{bmatrix} a_1 & -a_1 & & & & & \\ 0 & a_2 & -a_2 & & & & \\ & 0 & a_3 & -a_3 & & & \\ & & & ... & & & \\ & & & & ... & & \\ & & & & & 0 & a_{p-1} \end{bmatrix} \\ &= a_1a_2...a_{p-1}. \end{aligned}$$

(b). We have

$$\begin{aligned} \epsilon(D,e) &= \tau(D,u)\prod_{u\in v}(outdeg(u)-1)! \\ &= a_1a_2...a_{p-1}(a_1-1)!(a_1+a_2-1)!(a_2+a_3-1)!..(a_{p-2}+a_{p-1})!(a_{p-1})!. \end{aligned}$$

Exercise 10.2. *The solution of the number for d-ary de Bruijn sequence of degree n is similar to the solution of the number for 2-ary de Bruijn sequence of degree n. The method of counting*

d-ary de Bruijn sequence is to establish a correspondence between them and the Eulerian tour of the directed graph $D(n,d)$ of d-ary de Bruijn graphs of degree n.

The graph $D(n,d)$ has d^{n-1} vertices, i.e., d^{n-1} d-ary sequences of length $n-1$. A pair of vertices $(a_1a_2...a_{n-1}, b_1b_2...b_{n-1})$ form an edge of $D(n,d)$ if and only if $a_2a_3...a_{n-1} = b_1b_2...b_{n-2}$. In other words, the e is an edge if the last $n-2$ term of $init(e)$ is consistent with the first $n-2$ term of $fin(e)$. For any vertices $u \in v$, we have

$$outdeg(u) = indeg(u) = d, \quad (*a_2a_3...a_{n-1}, \text{ there are } d \text{ options available at } *).$$

So the D is balanced. The edge associated with each vertex is $2d$, with a total of d^{n-1} vertices. According to the handshake theorem, there are $\frac{d^{n-1}\cdot 2d}{2} = d^n$ edges. The $D(n,d)$ is connected. There is a one-to-one correspondence between the 2-ary de Bruijn sequence and the Eulerian tour of its corresponding de Bruijn graph. Similarly, there is also a one-to-one correspondence for the 2-ary de Bruijn sequence. For the graph $D(n,d)$, we first find $\tau(D,v)$ and then find $\epsilon(D,e)$

Lemma: If u and v are any two vertices of $D(n,d)$, then a directed path of length $n-1$from u to v exists and is unique.
Proof: Let $u = a_1a_2...a_{n-1}, v = b_1b_2...b_{n-1}$, then we have

$$u = a_1a_2...a_{n-1} \to a_2a_3...a_{n-1}*_1 \to a_3a_4...a_{n-1}*_1*_2 \to ... \to *_1*_2...*_{n-1}.$$

So we have $*_i = d_i$. This completes the proof.

Theorem: The eigenvalues of $L(D(n,d))$ are as follows: an eigenvalue of 0 with multiplicity one and an eigenvalue of d with multiplicity $d^{n-1}-1$.
Proof: Let A be a directed adjacency matrix. By lemma $A^{n-1} = J$ ($d^{n-1} \times d^{n-1}$' matrix of all 1's). The eigenvalues of J are as follows: an eigenvalue of d^{n-1} with multiplicity one and an eigenvalue of 0 with multiplicity $d^{n-1}-1$. Therefore, the eigenvalues of A are as follows: an eigenvalue of $d\zeta$ with multiplicity one and an eigenvalue of 0 with multiplicity $d^{n-1}-1$, where ζ is a specific $n-1$-degree primitive root of unity. By there are d rings for A, the trace of A is d. We have $\zeta = 1$. By $L(D) = dI - A$, The eigenvalues of L are as follows: an eigenvalue of 0 with multiplicity one and an eigenvalue of d with multiplicity $d^{n-1}-1$. This completes the proof.

The number of d-ary de Bruijn sequences of length n which begin with n 0's is

$$\begin{aligned}
\epsilon(D,e) &= \tau(D,u)\prod_{u\in v}(outdeg(u)-1)! \\
&= \frac{1}{d^{n-1}} \cdot d^{d^{n-1}-1}\prod_{u\in v}(outdeg(u)-1)! \\
&= d^{d^{n-1}-n} \cdot [(d-1)!]^{d^{n-1}}.
\end{aligned}$$

Exercise 10.3. .

Exercise 10.4. .

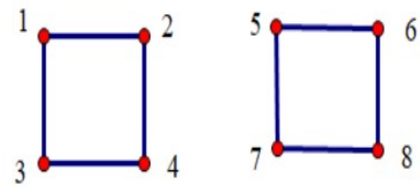

Figure 51:

Exercise 10.5. *The exercise lacks one condition: the connectivity of graph G. The Figure 51 is a 2-degree regular graph with 8 vertices, 8 edges, and no self-loops.*

(a). By the handshake theorem, we have $\frac{pd}{2} = q$, i.e., $pd = 2q$.

(b). Matrix A is a non negative matrix. The adjacency matrix A of graph G is irreducible $\Leftrightarrow$ G is connected and is not an isolated vertex. By Perron-Frobenius Theorem, the eigenvalue of A exists and is greater than 0, denote $\lambda_{\max}$. We consider the number of closed walks as follows. The number of walks with a length of 5 starting from v_i is d^ℓ. So the number of closed walks satisfies $\leqslant p \cdot d^\ell$. We have $\lambda_i \leqslant d$. (If not, there is $\lambda_i > 0$. Let $\ell \to \infty$. This is a contradiction.)

We consider the Laplacian matrix $L(G) = dI - A$ for G. The eigenvalues $d - \lambda_i$ of $L(G)$ are non negative. And the row and column sums of $L(G)$ are both 0. Therefore, there must be a 0 eigenvalue. So there is a $\lambda_{\max} = d$.

(c). If G has no multiple edges, then the number of closed walks of length 2 in G is $pd = 2q$.

(d). The number of closed walks of length l in G is $6^l + 2 \cdot (-3)^l$. If $l = 2$, then we have $pd = 2q = 36 + 2 \cdot 9 = 54$. So we have $q = 27$. By $pd = 54 = 2 \times 27 = 3 \times 18 = 6 \times 9$ ($pd = 1 \times 54$ is impossible) and G has not multiple edges and self-loops, we have $p > d$ and

d	p
2	27
3	18
6	9

.

By (b), we have $\lambda_i \leqslant d$. We know that $6 \leqslant d$ by the condition of (d). So the only possibility is $d = 6, p = 9, q = 27$. By

$$f_G(l) = \lambda_1^l + \lambda_2^l + ... + \lambda_p^l = 6^l + 2 \cdot (-3)^l,$$

the eigenvalues of G are $6, -3, -3, 0, 0, 0, 0, 0, 0$. The G is d degree regular, we have

$$\kappa(G) = \frac{1}{p}(d - \lambda_1)(d - \lambda_2)...(d - \lambda_{p-1}) = \frac{1}{9} \cdot 9 \cdot 9 \cdot 6^6 = 9 \cdot 6^6.$$

(e). For the number of closed walks in G walk along each edge of G exactly once in each direction, we need to find the number of Eulerian tour of the graph $\widehat{G}$. The G is a

connected undirected graph without self-loops. The $\widehat{G}$ is a directed graph that replaces each edge e ($\varphi(e) = \{u, v\}$) in G with a pair of directed edges e' and e'' that satisfy $\widehat{\varphi}(e') = (u, v)$ and $\widehat{\varphi}(e'') = (v, u)$. The $\widehat{G}$ is connected and balanced.

In here, for any $u \in V$, we have $outdeg(u) = d$. By Example 10.6, the Matrix-Tree Theorem is a corollary of Theorem 10.4. We have $L(G) = L(\widehat{G})$. Therefore we have

$$\tau(\widehat{G}, v) = detL_0(G) = detL_0(\widehat{G}) = \kappa(G).$$

By (d), we have $\kappa(G) = 9 \cdot 6^6$. We have

$$\epsilon(\widehat{G}, v) = \tau(\widehat{G}, v) \cdot \prod_{u \in V}(outdeg(u) - 1)! = 9 \cdot 6^6[(d-1)!]^p.$$

By (d), we have $p = 9, d = 6$ and $\epsilon(\widehat{G}, v) = 9 \cdot 6^6 \cdot (5!)^9$.

Exercise 10.6. (*a*). *Incomplete.........*

By $\epsilon(G, v) = \tau(G, v) \cdot \prod_{u \in V}(outdeg(u) - 1)!$, we have $outdeg(u) - 1 = 1$ or 0. We have $outdeg(u) = 1$ or 2 for any $u \in V$. We need to find that the number of balanced digraph with p vertices satisfy $outdeg(u) = 1$ or 2 and $\tau(D, v) = 1$.

(*b*)..........

Exercise 10.7. *The digraph D has p vertices and is $2d$ degree regular. For any $u \in V$, we have $outdeg(u) = indeg(v) = d$. The graph D' has p vertices and is $4d$ degree regular. For any $u \in V$, we have $outdeg(u) = indeg(v) = 2d$. By*

$$\epsilon(D, v) = \tau(D, v) \cdot \prod_{u \in V}(outdeg(u) - 1)! = det(L_0(D)) \cdot ((d-1)!)^p$$

and D' is the graph obtained from D by doubling each edge, we have $L(D') = 2L(D)$. We have

$$detL_0(D') = 2^{p-1}detL_0(D).$$

Therefore, we have

$$\begin{aligned}
\epsilon(D', e') &= \tau(D', v) \cdot \prod_{u \in V}(outdeg(u) - 1)! \\
&= det(L_0(D')) \cdot ((2d-1)!)^p \\
&= 2^{p-1}detL_0(D) \cdot ((2d-1)!)^p \\
&= 2^{p-1} \cdot \frac{[(2d-1)!]^p}{[(d-1)!]^p} \cdot \epsilon(D, e).
\end{aligned}$$

Exercise 10.8. (*a*). *Let ℓ be a fixed positive integer. We have $A^l = J$, where J is a matrix of order p of all 1's. The eigenvalues of J are as follows: an eigenvalue of p with multiplicity one and an eigenvalue of 0 with multiplicity $p-1$. The eigenvalues of A are as follows: an eigenvalue of $\sqrt[l]{p} \cdot \zeta$ with multiplicity one and an eigenvalue of 0 with multiplicity $p-1$, where ζ*

is a ℓ-degree primitive root of unity. (Note: If Ais a real symmetric matrix, then the eigenvalues of A must be real numbers. However, at this point, A is a directed adjacency matrix, and it is currently uncertain whether it is symmetric or not.) By $tr(A) = 0 \times (p-1) + \sqrt[l]{p} \cdot \zeta$ is a non-negative integer (The sum of diagonal elements of A). So we have $\zeta = 1$, $\sqrt[l]{p}$ is a positive integer. The eigenvalues of A are as follows: an eigenvalue of $\sqrt[l]{p}$ with multiplicity one and an eigenvalue of 0 with multiplicity $p-1$.

(b). The number of self-loops in D is equal to the sum of diagonal elements in the directed adjacency matrix of D. There are $\sqrt[l]{p}$ self-loops in D.

(c). Incomplete.....

For any vertices u, v of D, there is unique directed walk of length ℓ form u to v. The D is connected. Let E be the column eigenvector corresponding to the maximum eigenvalue of $A(D)$. We have $A \cdot E = \sqrt[l]{p} \cdot E$. By $(A \cdot E)^t = (\sqrt[l]{p} \cdot E)^t$, we have $E^t \cdot A^t = \sqrt[l]{p} \cdot E^t$. We have

$$E^t \cdot A^t \cdot E = \sqrt[l]{p} \cdot E^t \cdot E = E^t(\sqrt[l]{p} \cdot E).$$

The A is a directed adjacency matrix of D. The A is non-negative. By D is connected, A is an irreducible matrix. By Perron-Frobenius Theorem, there is an eigenvector for $\sqrt[l]{p}$ (unique up to multiplication by a positive real number) all of whose entries are positive. Let E be this eigenvector. By $E^t(A^tE - \sqrt[l]{p}E) = 0, E^t = (*_1, *_2, ..., *_p), *_i > 0,\ 1 \leqslant i \leqslant p$, we have $A^tE = \sqrt[l]{p}E = AE$. (It's not over yet, for example: $(-1, 1)(1, 1)^t = 0$). We need to prove

$$A \begin{pmatrix} 1 \\ 1 \\ .. \\ 1 \end{pmatrix} = A^t \begin{pmatrix} 1 \\ 1 \\ .. \\ 1 \end{pmatrix}.$$

(d).......... Incomplete

If we have $outdeg(u) = indeg(u) = d$ for any $u \in V$, then the number of direct walk of length l from v_1 is d^l for any $v_1 \in V$. We know that the existence and uniqueness of directed walks with length l from v_1 to any vertex. There are p vertices in D. We have $d^l = p$.

(e). The eigenvalues of $A(D)$ are as follows: an eigenvalue of $\sqrt[l]{p} = d$ with multiplicity one and an eigenvalue of 0 with multiplicity $p-1$. We have $L(D) = dI - A(D)$. The eigenvalues of $L(D)$ are as follows: an eigenvalue of 0 with multiplicity one and an eigenvalue of d with multiplicity $p-1$. Therefore we have $\tau(D, v) = \frac{1}{p} \cdot d^{p-1}$ and

$$\epsilon(D, v) = \tau(D, v) \cdot \prod_{u \in V} (outdeg(u) - 1)! = \frac{1}{p} \cdot d^{p-1}[(d-1)!]^p.$$

(f).....

Exercise 10.9. *(a). If $n \geqslant 3$, then we have $n! \geqslant 2n$. There are $n!$ permutations of $[n]$. There are $n!$ circular factor of length n for sequence $a_1a_2...a_{n!}$. Suppose this sequence (in Exercise)* *exist. The* $123...n$ *is a permutation of* $[n]$. *Let* $12...n$ *be the first* n *terms for this sequence under modular* $n!$ *operation. We have*

$$123...n\underline{*_1}\ \underline{*_2}...\underline{*_n}......$$

The sequence $*_1 = 1$ *in the* $n+1$*-term. If not, the circular factors* $23...n*_1$ *starting* 2 *is not the permutation of* $[n]$. *Similarly, we have* $*_2 = 2, *_3 = 3, ..., *_n = n$. *However,* $*_1 *_2 ... *_n = n$ *is same as* $12...n$. *(By* $n! \geqslant 2n$, *there exist* $*_1 *_2 ... *_n = n$*). This is a contradiction. There does exist this sequence.*

(b).........

(c). By $n = 3$, *the all permutation of* $n-1$ *terms are* $12, 13, 21, 23, 31, 32$. *By* $n! = 6$ *and* $123 *_1 *_2 *_3$, *the* $*_1$ *is* 1 *or* 2. *If* $*_1 = 1$, *the only possibility is* $*_2 = 3, *_3 = 2$. *If* $*_2 = 2$, *the only possibility is* $*_2 = 1, *_3 = 3$. *The number of universal cycles beginning with* 123 *is 2, i.e.,* 123132, 123213.

Note: In fact, there are 3 universal cycles. The other one is 121323 (in an equivalent sense).

(d)..........

11 Cycles, Bonds, and Electrical Networks

Exercise 11.1. *(a). Each vertex in* C_n *is connected to* n *vertices. By handshake theorem, there are* $\frac{2^n \cdot n}{n} = 2^{n-1} \cdot n$ *edges in* C_n. *By*

$$dim\beta = p - k = 2^n - 1,$$

we have

$$dimC = q - p + k = 2^{n-1} \cdot n - 2^n + 1 = 2^{n-1}(n-2) + 1.$$

By Lemma 11.4, $0 \neq f \in C$, $||f||$ *contains an undirected circle. The* C_n *is a bipartite graph, which only has even cycles. Three edges cannot form a circuit in* C_n. *Therefore, there is not circulation supported on three edges.*

(b)..........

Exercise 11.2. .

Exercise 11.3. ....

Exercise 11.4. *We have bonds* $\Leftrightarrow$ *minimal cut-set. Let set* A *be a set that satisfies the requirements of the question. The* A *is not contain bonds and the largest set of edges. The* $K_p \setminus A$ *is also connected. If* A *contains the most elements, then* $K_p \setminus A$ *is a tree. Therefore we have*

$$\#A = \frac{p(p-1)}{2} - (p-1) = \frac{(p-2)(p-1)}{2}.$$

The A *can contain up to* $\frac{(p-2)(p-1)}{2}$ *edges. The number of this set is* $\kappa(K_p) = p^{p-2}$.

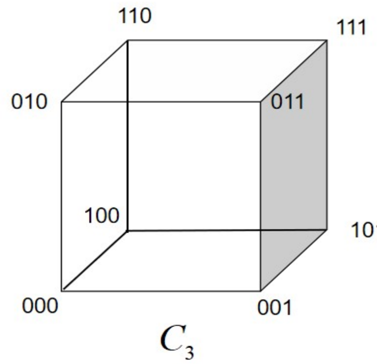

Figure 52:

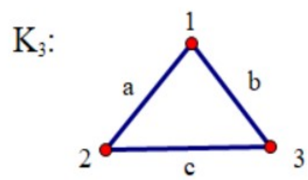

Figure 53:

For example, for K_3*, we have* $A = \{a\}, \{b\}, \{c\}$ *and* $p^{p-2} = 3^{3-2} = 3$*. For* K_4*, we have*

$$
\begin{aligned}
A = &\{acd\}, \{cde\}, \{ace\}, \{ade\},\\
&\{cdf\}, \{cdb\}, \{bcf\}, \{bdf\},\\
&\{acf\}, \{def\}, \{bde\}, \{abd\},\\
&\{abe\}, \{bef\}, \{afe\}, \{abf\},
\end{aligned}
$$

and $p^{p-2} = 4^2 = 16$.

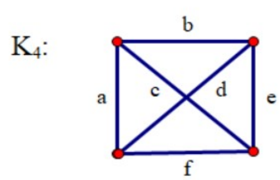

Figure 54:

Exercise 11.5. .

Exercise 11.6. .

Exercise 11.7. (*a*). *Counterexample. The edge-transitive graph is not vertex-transitive: star graph in Figure 55.*

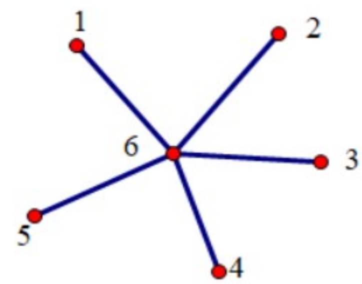

Figure 55:

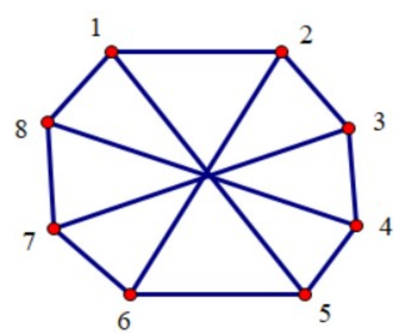

Figure 56:

The vertex-transitive graph is not edge-transitive: Figure 56. For Figure 56, this is vertex-transitive graph, but this is not edge-transitive. For the edge $1-2$, there is only one circuit including $1-2$ of length 4, i.e., $1-2-6-5-1$. For the edge $1||5$, there are 2 circuit including $1-5$ of length 4, i.e., $1-5-6-2-1$ and $1-5-4-8-1$. there is not an automorphism which takes $1-2$ to $1-5$. So the graph is not edge-transitive.

(b). We have

$$R(D) = -\frac{1}{I_q} = \frac{\text{the number of spanning trees containing edge } e}{\text{the number of spanning trees not containing edge } e}.$$

Let x be the number of spanning trees containing edge e. Let y be the number of spanning trees not containing edge e. Then we have $R(D) = \frac{x}{y}$. Let the q edges be $e_1, e_2, ..., e_q$. Let x_i be the number of spanning trees containing edge e_q. By the graph G is edge-transitive, we have $x_1 = x_2 = ... = x_q$. A tree containing $p-1$ edges. Therefore, we have

$$\frac{x \cdot q}{p-1} = \kappa(G) = x + y,$$

and $\frac{x}{y} = \frac{p-1}{q-p+1}$.

Exercise 11.8. *(a). Let D be a loopless connected digraph with q edges. If there are p vertices in D, then we have $dim\mathcal{B} = p-1$ and $dim\mathcal{C} = q-p+1$. By Lemma 11.6, the columns of $B[S]$ are linearly independent if and only if S is acyclic. By $dim\mathcal{B} = p-1$, the S is a no circuit graph with $p-1$ edges. Thus S is a tree. By Binet-Cauchy Theorem,*

$$detBB^t = \sum_S (detB[S])(detB^t[S]),$$

where S ranges over all $p-1$-element subsets of the edges of D, and $A^t[S] = A[S]^t$, we have

$$detBB^t = \sum_S (detB[S])^2.$$

By Theorem 11.13, the basis matrices B of $\mathcal{B}$ are unimodular. When S is an edge of a tree, we have $detB[S] = \pm 1$, other cases are 0. At this time, in the right-hand side of the above equation, when S forms the edge set of a spanning tree of D, it is equal to 1, and in other cases it is 0. Therefore we have $detBB^t = \kappa(D)$.

Similarly, By Theorem 11.6, the columns of $C[S]$ are linearly independent if and only if S contains no bond. By the proof of Theorem 11.13, $detC_T[T_1^*] = \pm 1$ if and only if T_1^* is a cotree. Therefore we have

$$detCC^t = \sum_S (detC[S])(detC^t[S]) = \sum_t (detC[S])^2,$$

where S ranges over all $q-p+1$-element subsets of the edges of D. If S is a cotree, then $detC[s] = \pm 1$ and other cases is 0. Then we have $detCC^t = \kappa(D)$.

(b). By $dimB = p-1, dimC = q-p+1$, we have $Z = \begin{pmatrix} C \\ B \end{pmatrix}$ is a $q \times q$ matrix and $Z^t = (C^t, B^t)$. We have

$$Z \cdot Z^t = \begin{pmatrix} C \\ B \end{pmatrix} (C^t \; B^t) = \begin{pmatrix} CC^t & CB^t \\ BC^t & BB^t \end{pmatrix}.$$

By the row vector of B is orthogonal to the row vector of C, we have $CB^t = 0, BC^t = 0$. We have

$$detZ \cdot Z^t = detCC^t \cdot detBB^t = (\kappa(D))^2.$$

by $detZ \cdot Z^t = (detZ)^2$, we have $detZ = \pm\kappa(D)$.

Exercise 11.9. .

Exercise 11.10. *We have*

$$R(D) = -\frac{1}{I_q} = \frac{\textit{the number of spanning trees containing edge } e_q}{\textit{the number of spanning trees not containing edge } e_q} = \frac{\#[e_q \in T]}{\#[e_q \notin T]}.$$

We have

$$R(D^*) = -\frac{V_q^*}{I_q^*} = V_q^*(-\frac{1}{I_q^*}) = V_q^* \frac{\#[e_q^* \in T]}{\#[e_q^* \notin T]}.$$

By Proposition 11.18, the set S is the set of edges of a spanning tree of $D \Leftrightarrow S^* = \{e^*, e \in S\}$ is the set of edges of a cotree of D^*, $e \in T \Leftrightarrow e^* \notin T$. We have

$$\#[e_q \in T] = \#[e_q^* \notin T], \#[e_q \notin T] = \#[e_q^* \in T],$$

and

$$R(D^*) = V_q^* \cdot \frac{1}{R(D)} = \frac{V_q^*}{R(D)}.$$

Exercise 11.11. *Let add edge e_1. By the definition of the extended Smith diagram of a squared rectangle, we have*

$$\begin{aligned} I :&E \to R \\ &I_e \mapsto \textit{The edge length of the square corresponding to edge } e \\ &I_{e_1} \mapsto -a \end{aligned}$$

$$\begin{aligned} V :&E \to R \\ &V_e \mapsto \textit{The edge length of the square corresponding to edge } e \\ &V_{e_1} \mapsto -a \end{aligned}$$

Therefore we have

$$< I, V >= \sum_{e \in E} I(e)V(e) = \sum_{e \in E \setminus e_1} I(e)V(e) - ab = \sum_{e \in E \setminus e_1} I^2(e) - ab = 0.$$

"geometric significance": In the squared rectangle, the sum of the areas of all internal small squares is equal to the sum of the areas of the rectangles.

Exercise 11.12. *Let D be the extended Smith diagram of $a \times b$ squared rectangle. For the new edge e_1 connecting two polar, let $I_{e_1} = b$ $V_{e_1} = -a$. The two Kirchhoff's Law are true for all vertices. For edge $R_e = 1$ outside e_1 , we consider this graph as a planar electrical network. By Corollary 11.15, we have*

$$R(D) = \frac{\textit{the number of spanning trees containing edge } e_1}{\textit{the number of spanning trees not containing edge } e_1} = \frac{\#[e_1 \in \kappa(D)]}{\#[e_1 \notin \kappa(D)]}.$$

By $R(D) = -\frac{V_{e_1}}{I_{e_1}} = \frac{a}{b}$, there exists a positive integer k such that

$$\#[e_1 \in \kappa(D)] = ka,\ \#[e_1 \notin \kappa(D)] = kb,\ \kappa(D) = ka + kb = k(a+b).$$

We have $(a+b)|\kappa(D)$.

12 A Glimpse of Combinatorial Commutative Algebra

Exercise 12.1. .

Exercise 12.2. .

Exercise 12.3. .

Exercise 12.4. .

Exercise 12.5. .

Exercise 12.6. .

Exercise 12.7. .

Exercise 12.8. .

Exercise 12.9. .

Exercise 12.10. .

Exercise 12.11. .

Exercise 12.12. .

Exercise 12.13. .

Exercise 12.14. .

Exercise 12.15. .

Exercise 12.16. .

Exercise 12.17. .

Exercise 12.18. .

Exercise 12.19. .

Exercise 12.20. .

Exercise 12.21. .

Exercise 12.22. .

Exercise 12.23. .

Exercise 12.24. .

Exercise 12.25. .

Exercise 12.26. .

Exercise 12.27. .

Exercise 12.28. .

Exercise 12.29. .

Exercise 12.30. .

Exercise 12.31. .

Exercise 12.32. .

13 Miscellaneous Gems of Algebraic Combinatorics

Exercise 13.1. *In Section 13.1, if $\alpha = \frac{1}{2}$, the strategy have a 30% chance of success in the limit as $n \to \infty$. If we want a 50% chance of success, we need $\alpha > \frac{1}{2}$. We have $2\alpha n > n$. Using the winning strategy in section 12.1, the success rate is*

$$1 - \sum_{r=2\alpha n+1}^{2n} \frac{1}{r} = 1 - \sum_{r=1}^{2n} \frac{1}{r} + \sum_{r=1}^{2\alpha n} \frac{1}{r}.$$

By

$$\lim_{n\to\infty} \left(\sum_{r=1}^{n} \frac{1}{r} - \log \alpha \right) = \gamma,$$

where γ is Euler's constant, the success rate (when $n \to \infty$) is

$$\begin{aligned} \lim_{n\to\infty} \left(1 - \sum_{r=2\alpha n+1}^{2n} \frac{1}{r}\right) &= \lim_{n\to\infty} \left(1 - \sum_{r=1}^{2n} \frac{1}{r} + \sum_{r=1}^{2\alpha n} \frac{1}{r}\right) \\ &= \lim_{n\to\infty} (1 - \log 2n + \log 2\alpha n) \\ &= 1 - \log \frac{1}{\alpha} = 1 + \log \alpha = 50\%. \end{aligned}$$

We have $log\alpha = -0.5$ and $\alpha = \frac{1}{\sqrt{e}} \approx 0.6065$. Therefore we have a 50% chance of success in the limit as $n \to \infty$ if $\alpha = \frac{1}{\sqrt{e}}$.

Exercise 13.2. *Incomplete.........*

A strategy: Similar to Section 13.1, the prisoners assign themselves the number $1, 2, 3, ..., 100$ *by whatever method they prefer. Each prisoner is assigned a different number. They regard the boxes, which are lined up in a row, as bing numbered* $1, 2, ..., 100$ *form left to right. The prisoner with the number* k *opens all boxes except for box* k*, totaling* 99*. A strategy can succeed because it aims to make all prisoners as not sees his or her name as possible, with each prisoner having to open* 99 *boxes.*

The probability is $\frac{1}{100!}$.

Exercise 13.3. (a). *If all prisoners randomly select a color at the same time. Then its success probability is* $\frac{1}{2^{100}}$. *Because the probability of each person guessing the color of their hat correctly each time is* $\frac{1}{2}$*, the success probability is* $\frac{1}{2^{100}} \leqslant p \leqslant \frac{1}{2}$. *Let red hat be 0, blue hat be 1. Record the number corresponding to the color of each prisoner as* $a_1, a_2, ..., a_{100}$*,* $a_i \in \{0, 1\}$*. Let*

$$S = a_1 + a_2 + ... + a_{100}.$$

So the sum of the numbers seen by a_i *is* $S - a_i$*. So* a_i *guessed that the color of his hat was red (when* $-(S - a_i) \mod 2 = 0$*); blue (when* $-(S - a_i) \mod 2 = 1$*). When* $S \equiv 0(\mod 2)$*, they guess the color of their hats correctly at the same time. When* $S \equiv 1(\mod 2)$*, they also guess the color of their hats incorrectly. So we have* $p = \frac{1}{2}$.

(b)*............*

Exercise 13.4. .

Exercise 13.5. (a). *Similar to Theorem 13.2. Let* k *be the number of clubs. Define a incidence matrix* $M = (M_{ij})$ *over the two-element filed* F_2*. The rows of* M *are indexed by the clubs* C_i *and the columns by the inhabitants* x_j *of Oddtown. Set*

$$M_{ij} = \begin{cases} 1, & x_j \in C_i \\ 0, & otherwise. \end{cases}$$

Then the sum of a row of M *is even. Let* $A = MM^t$*, a* $k \times k$ *matrix. Every club has an even number of members, the main diagonal elements of* A *are 0. Every club has an odd number of members, the main diagonal elements of* A *are 1. Therefore we have* $A = J - I$*, where* J *is a* $k \times k$ *matrix of all* 1*'s and* I *is a* $k \times k$ *identity matrix. We have*

$$n \geqslant rank(M) \geqslant (MM^t) = rank(A) = k.$$

The maximum number of clubs is n.

(b)*..........*

Exercise 13.6. (a). *The author believes that the title is incorrect. The maximum number of clubs should be* $2^{\lfloor \frac{n}{2} \rfloor} - 1$.
For example: The clubs that meet the conditions for a, b, c *are* $\{a, b\}$*. However, we have* $2^{\lfloor \frac{3}{2} \rfloor} =$

2. *The clubs that meet the conditions for* a, b, c, d, e *are* $\{a, b\}, \{c, d\}, \{a, b, c, d\}$. *However, we have* $2^{\lfloor \frac{5}{2} \rfloor} = 4$.

Proof: Suppose that n *people live in Eventown. Every club contains an even number of persons. The smallest non-zero even number is 2. These* n *people can form* $\lfloor \frac{n}{2} \rfloor$ *clubs, with two people in each club. These* $\lfloor \frac{n}{2} \rfloor$ *clubs have no contact with each other. Let* $m = \lfloor \frac{n}{2} \rfloor$. *For the club with 4 people, there are* $\binom{m}{2}$. *For the club with 6 people, there are* $\binom{m}{3}$. *And so on. The maximum number of clubs is*

$$\binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \cdots + \binom{m}{m} = 2^m - 1 = 2^{\lfloor \frac{n}{2} \rfloor} - 1.$$

(b)..........

Exercise 13.7. *Define two incidence matrices* $M = (M_{ij}), N = (N_{ij})$ *over the two-element field* F_2. *The rows of* M *and* N *are indexed by the clubs* $R_1, R_2, ..., R_m$ *and* $B_1, B_2, ..., B_m$, *respectively. The columns by the inhabitants* x_j. *Set*

$$M_{ij} = \begin{cases} 1, & x_j \in R_i \\ 0, & otherwise. \end{cases}$$

and

$$N_{ij} = \begin{cases} 1, & x_j \in B_i \\ 0, & otherwise. \end{cases}$$

For any i, $\#(R_i \cap B_i)$ *is odd number. The main diagonal elements of* $M \cdot N^t$ *is 1. For* $i \neq j$, $\#(R_i \cap B_j)$ *is even number. The off-diagonal elements of* $M \cdot N^t$ *is 0. So* $M \cdot N^t = I_{m \times m}$. *We have*

$$m = rank(M \cdot N^t) \leqslant rank(M) \leqslant n.$$

This completes the proof.

Exercise 13.8. *Define a incidence matrices* $M = (M_{ij})$ *over the three-element field* F_3. *The rows of* M *are indexed by the clubs* $C_1, C_2, ..., C_k$. *The columns by the inhabitants* x_j. *Set*

$$M_{ij} = \begin{cases} 1, & x_j \in C_i \\ 0, & otherwise. \end{cases}$$

We consider the $k \times k$ *matrix* MM^t. *Any two different clubs have a multiple of 3 public members. So the off-diagonal elements of* MM^t *are 0 in* F_3. *There are* $j(\leqslant k)$ *elements that is not 0 in diagonal elements. We have*

$$j = rank(M \cdot M^t) \leqslant rank(M) \leqslant n.$$

This completes the proof.

Exercise 13.9. *Define a incidence matrices $M = (M_{ij})$ over the two-element field F_2. The rows of M are indexed by the clubs $C_1, C_2, ..., C_k$. The columns by the inhabitants x_j. Set*

$$M_{ij} = \begin{cases} 1, & x_j \in C_i \\ 0, & otherwise. \end{cases}$$

We consider the $k \times k$ matrix MM^t. We have

$$M \cdot M^t = \begin{pmatrix} & & & & 1 \\ & 0 & & 1 & \\ & & & \cdot\cdot & \\ & & \cdot\cdot & & \\ & 1 & & 0 & \\ 1 & & & & \end{pmatrix}_{k\times k}.$$

So $k = rank(M \cdot M^t) \leqslant rank(M) \leqslant n$. This completes the proof.

Exercise 13.10. *Define a incidence matrices $M = (M_{ij})$ over the two-element field F_2. The rows of M are indexed by the clubs $C_1, C_2, ..., C_k$. The columns by the inhabitants x_j. Set*

$$M_{ij} = \begin{cases} 0, & x_j \in C_i \\ 1, & otherwise. \end{cases}$$

So

$$M \cdot M^t = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & & \cdots & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}_{k\times k}.$$

We have

$$rank(MM^t) = \begin{cases} k, & k \ \ is \ \ even \\ k-1, & k \ \ is \ \ odd. \end{cases}$$

We have $rank(M \cdot M^t) \leqslant rank(M) \leqslant n$. As a function of n, when k is even, the maximum possibility of k is the maximum even number of $\leqslant n$, and when k is odd, the maximum possibility of k is the maximum odd number of $\leqslant n+1$.

Exercise 13.11. (a). *We have $|A| = |A^t| = |-A| = (-1)^n|A|$. If n is odd, then we have $|A| = -|A|$ and $|A| = 0$.*

(b)*..........Perhaps related to odd neighborhood covers.*

Exercise 13.12. *We consider the eigenvalues of skew symmetric matrices. Let A be a skew symmetric matrix. The eigenvalues of A are $\lambda = a + bi$. The corresponding eigenvectors is*

$x = u+vi \neq 0$, where u, v are non-zero real vectors. So $Ax = \lambda x$ and $A(u+vi) = (a+bi)(u+vi)$, i.e.,

$$Au + iAv = (au - bv) + (bu + av)i.$$

We have

$$\begin{cases} Au = au - bv, \\ Av = bu + av. \end{cases}$$

Thus

$$u^T Au = au^T u - bu^T v,\ v^T Av = bv^T u + av^T v.$$

Therefore we have

$$u^T Au + v^T Av = au^T u + av^T v = a(|u|^2 + |v|^2).$$

By $u^T Au$ is a number, we have

$$u^T Au = (u^T Au)^T = u^T A^T u = -u^T Au \Rightarrow u^T Au = -u^T Au.$$

So $u^T Au = 0$. Similarly, we have $v^T Av = 0$. We have

$$u^T Au + v^T Av = a(|u|^2 + |v|^2) = 0.$$

By

$$u + vi \neq 0 \Rightarrow |u|^2 + |v|^2 \neq 0 \Rightarrow a = 0,$$

we have $\lambda = bi,\ b \in R$. Therefore, the eigenvalues of a skewed symmetric matrix are either 0 or pure imaginary numbers. Let V' be the subspace of M_n composed of all skew symmetric matrices. We have $dim(V \cap V') = 0$. So

$$dim V' = (n-1) + (n-2) + ... + 1 = \frac{(n-1)n}{2}.$$

Therefore we have

$$dim V \leqslant dim M_n - dim V' = n^2 - \frac{n(n-1)}{2} = \binom{n+1}{2}.$$

Exercise 13.13. *For the proof of Theorem 13.3, we restricted it within the two-element field F_2. We have*

$$E(K_n) = E(B_1) \cup E(B_2) \cup ... \cup E(B_m).$$

The B_k is a complete bipartite graph with vertex bipartition (X_k, Y_k). Define matrix A_k by

$$(A_k)_{ij} = \begin{cases} 1, & i \in X_k, j \in Y_k \\ 0, & otherwise. \end{cases}$$

Each edge of K_n is covered an odd number of times. We have $S + S^t = J - I$ in F_2. Then we have

$$m \geqslant rank S \geqslant n - 1 \geqslant \frac{n-1}{2}.$$

Note: The meaning of the question should be that there exists a complete bipartite partition of $m = \frac{n-1}{2}$.

Exercise 13.14. *Incomplete.............*

Conjecture: the minimum value of m is $\lfloor \frac{n}{2} \rfloor$.

A complete tripartite graph can allow for a X_i is an empty set, in which case it is a complete bipartite graph. By Theorem 13.3, m can take $n-1$, but m should have a smaller lower bound. Let x a smaller lower bound. Then..............

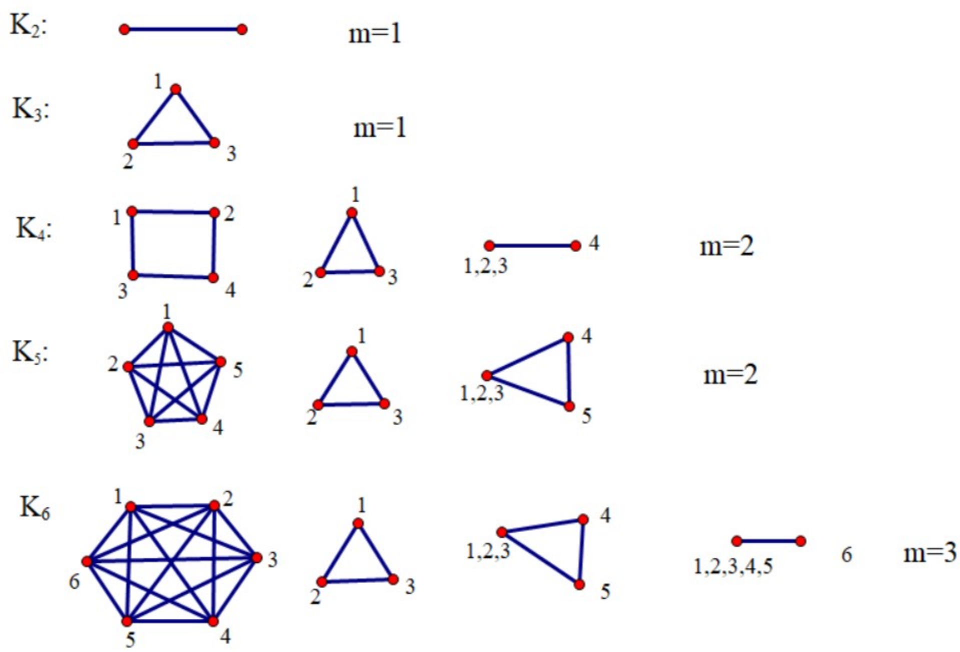

Figure 57:

Similar to Figure 57, we have $m = 3$ for K_7, $m = 4$ for K_8, and $m = 4$ for K_9. So m possible reach $\lfloor \frac{n}{2} \rfloor$.

Exercise 13.15. *(Non algebraic proof): Define a $n \times n$ incidence matrices $M = (M_{ij})$. The rows of M are indexed by $A_1, A_2, ..., A_n$. The columns by $x_1, x_2, ..., x_n$. Set*

$$M_{ij} = \begin{cases} 1, & x_j \in A_i \\ 0, & \textit{otherwise.} \end{cases}$$

The $A_i - \{x\}$ means deleting the column where x is located in M. We use "Mathematical induction". If $n = 2$, A_1 and A_2 is different from each other, there must be an element in row A_1 that is different from the same column element in row A_2. Then we reserve this column and delete another column. A_1 and A_2 still have different rows. Suppose the proposition is hold for $n \times n$ matrix M. We consider the $(n+1) \times (n+1)$ incidence matrices M. Now we remove the elements from the first column. If each row is different from each other, the conclusion holds. If there are identical rows, only two rows can be the same, denoted A_i, A_j. (If the three rows A_i, A_j, A_k are the same, and the first column element is 0 or 1, then there must be two *rows that are the same in the original matrix, which is contradictory). Now we delete A_i row, then a $n \times n$ matrix is obtained. According to the inductive assumption, there exists a column, which can be set as the last column. After we delete it, the remaining n rows are different from each other. At this point, we delete the last column from the original $(n+1) \times (n+1)$ matrix. By $A_i - x = A_j - x$, the rows where A_i and A_j are located are different from the remaining $n-1$ rows. The row where A_i and A_j are located in the original matrix that the first column being 0 and the other being 1, so they are also different from each other. Therefore, even after deleting the last column, all rows are still different from each other.*

(Algebraic proof)...............

Exercise 13.16. .

Exercise 13.17. .

Exercise 13.18. *Let H be a Hadamard matrix. If $n = 1, 2$, we have the Hadamard matrix as follow*

$$(1), \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Now, suppose $n \geqslant 3$ and $H = (h_{ij})$. By $n \geqslant 3$ and $H \cdot H^t = nI$, we have

$$\begin{pmatrix} h_{11} & h_{12} & . & . & . & h_{1n} \\ h_{21} & h_{22} & . & . & . & h_{2n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ h_{n1} & h_{n2} & . & . & . & h_{nn} \end{pmatrix} \begin{pmatrix} h_{11} & h_{21} & . & . & . & h_{n1} \\ h_{12} & h_{22} & . & . & . & h_{n2} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ h_{1n} & h_{2n} & . & . & . & h_{nn} \end{pmatrix} = \begin{pmatrix} n & & & & & \\ & n & & & & \\ & & . & & & \\ & & & . & & \\ & & & & . & \\ & & & & & n \end{pmatrix}$$

Therefore we have

$$\sum_{j=1}^{n} h_{1j}^2 = n,$$

and

$$\sum_{j=1}^{n} h_{1j}^2 = \sum_{j=1}^{n} (h_{1j}^2 + h_{1j} \cdot h_{3j} + h_{2j} \cdot h_{1j} + h_{2j} \cdot h_{3j}) = \sum_{j=1}^{n} (h_{1j} + h_{2j})(h_{1j} + h_{3j}).$$

We have

$$\sum_{j=1}^{n} (h_{1j} + h_{2j})(h_{1j} + h_{3j}) = n.$$

For any j, we have $h_{1j} + h_{2j} = 0, \ \pm 2; \ h_{1j} + h_{3j} = 0, \ \pm 2$. Therefore, the sum of each term on the left in the above equation is a multiple of 4. We have $4|n$.

Exercise 13.19. .

Exercise 13.20. .

Exercise 13.21. .

Exercise 13.22. .

Exercise 13.23. *Incomplete........*

Let $f(n)$ be the number of paths form $(0,0)$ to (n,n), where each step in the path is $(1,0)$, $(0,1)$ or $(1,1)$. If we use HVD paths with r diagonal steps, then we must have used $n-r$ horizontal steps and $n-r$ vertical steps, where H is the horizontal step, V is the vertical step and D is horizontal step. So the number of HVD paths with r diagonal steps was used that is the number of permutations of the multiple set $\{(n-r)\cdot H, (n-r)\cdot V, r\cdot D\}$. Therefore we have

$$f(n) = \sum_{r=0}^{n} \frac{(2n-r)!}{((n-r)!)^2 \cdot r!} = \sum_{r=0}^{n} \binom{2n-r}{n-r \quad n-r \quad r}.$$

Exercise 13.24. (a). *Let $g(n) = \alpha_1^n$. Then we have $g(n) - \alpha \cdot g(n-1) = 0$ and $g(n) = \alpha_1^n \in \mathcal{P}$. By Q_1^n is nonzero complex polynomial, we have*

$$Q_1(n) = c_q n^q + c_{q-1} n^{q-1} + ... + c_1 n + c_0, \ c_i \in \mathcal{C}.$$

For n^q, we have $n^q - n \cdot q^{n-1} = 0$. So $n^q \in \mathcal{P}$. By Theorem 13.18, we have $Q_1(n) = \mathcal{P}$. So $Q_1(n)\alpha_1^n \in \mathcal{P}$. Once again by Theorem 13.18, we have $f(n) \in \mathcal{P}$.

(b)............

Exercise 13.25. .

Exercise 13.26. .

Exercise 13.27. .

Exercise 13.28. .

Exercise 13.29. .

Exercise 13.30. .

Exercise 13.31. .

Exercise 13.32. .

Exercise 13.33. .

Exercise 13.34. .

Exercise 13.35. .

Exercise 13.36. .

Exercise 13.37. .

Exercise 13.38. .

Exercise 13.39. .

Exercise 13.40. .

Exercise 13.41. .

Exercise 13.42. .

Exercise 13.43. .

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